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Error Analysis of the Supplementary Variable Method for the Allen-Cahn Model

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Abstract. The supplementary variable method has received much attention as a new technique to construct structure-preserving algorithm that can preserve the original energy law. However, there has been a lack of relevant convergence analysis results, mainly because the method requires solving a nonlinear algebraic equation for the supplementary variable, and the non-uniqueness of its root makes estimating the root's error very difficult. In this paper, we take the Allen-Cahn model as an example and construct a second-order scheme using the supplementary variable method that preserves the original energy dissipation law. We then give a sufficient condition to ensure that the root of the nonlinear algebraic equation exists uniquely in the neighborhood of its exact solution. Under this condition, and when the time step is sufficiently small, we establish a rigorous error estimator for this scheme. Finally, we validate the effectiveness of the proposed scheme through several numerical examples.

AMS subject classifications: 35Q99, 65M06, 65M12

Key words: Allen-Cahn model, supplementary variables method, second order, original energy dissipation law, H^2 norm error estimate.

1 Introduction

The Allen-Cahn (AC) model is an effective method for describing the coarsening dynamics of a binary alloy system [1]. Consider a free energy functional of the Ginzburg-Landau

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type,

$$E[\Phi]=\int_\Omega\left[\frac{\epsilon^2}{2}|\nabla\Phi|^2+F(\Phi)\right]\mathrm{d}\boldsymbol{x}\quad\text{with}\quad F(\Phi):=\frac{1}{4}(\Phi^2-1)^2,\tag{1.1}$$

where $\boldsymbol{x}\in\Omega\subseteq\mathbb{R}^2$ and $0<\epsilon<1$ is a bounded parameter proportional to the interface width. Then the Allen-Cahn model would be given by the L^2 gradient flow associated with the free energy functional $E[\Phi]$,

$$\partial_t\Phi=-\kappa\mu\quad\text{with}\quad\mu:=\frac{\delta E}{\delta\Phi}=f(\Phi)-\epsilon^2\Delta\Phi,\quad f(\Phi)=\Phi^3-\Phi,\tag{1.2}$$

where the parameter κ is the mobility related to the characteristic relaxation time of the system and μ represents the chemical potential. Assume that Φ is periodic over the domain Ω, the following energy dissipation law can be derived

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\kappa\|\mu\|^2\leq 0,\quad\text{for } t>0,\tag{1.3}$$

where $\langle u,v\rangle:=\int_\Omega uv\mathrm{d}\boldsymbol{x}$, and the associated L^2 norm $\|v\|=\sqrt{\langle v,v\rangle}$ for all $u,v\in L^2(\Omega)$.

The gradient flow problem has attracted much attention in the past decades, and many mature and effective numerical methods have been used to solve it. These include the convex splitting method, which constructs a uniquely solvable and energy-stable numerical scheme through splitting the energy [5, 6, 12]. The stabilizer technique achieves unconditional energy stability by introducing a regularization term [13, 18]. Although these methods satisfy the property of energy decay, they do not preserve the actual energy dissipation law, which may cause non-physical dissipation, either enhanced or weakened and thereby lose accuracy in tracking dynamics of the system. To address this problem, alternative methods have emerged, such as Invariant Energy Quadratization (IEQ) [16, 25] and Scalar Auxiliary Variables (SAV) [23]. These methods are based on quadraticizing the energy and introducing auxiliary variables to create efficient energy stabilization schemes, while preserving the energy dissipation law of the system. However, it should be noted that these methods only preserve an energy dissipation law for a modified free energy, not the original energy.

In recent years, some efficient and stable methods have been developed to preserve the original energy dissipation law. These include traditional projection methods [4, 11], and an implicit energy preservation method developed for the Cahn-Hilliard model [27], in which the original energy dissipation law is strictly preserved. Shen et al. [7] proposed a new Lagrange multiplier method preserving the original energy dissipation law, which is based on an extended PDE system consisting of the original PDE system and its energy equation. Inspired by this, Gong et al. [3, 14, 19] proposed the supplementary variable method (SVM), which constructs the extended PDE system consisting of the original PDE system and its energy dissipation law, which naturally achieves the preservation of the original energy dissipation law and has more freedom in the choice