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Modelling and Numerical Shape and Topology Optimization for a Hemivariational Inequality in Stokes Flows with Slip Boundary Conditions

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Abstract. This paper focuses on modelling and numerical shape and topology optimization for a nonlinear hemivariational inequality in a Stokes flow with contact and frictional conditions. For the energy functional to be minimized subject to geometric volume constraint, we present shape sensitivity analysis and propose a volume preserving gradient-type method for shape optimization. Moreover, we build a relaxed topology optimization model for the hemivariational inequality in a Stokes fluid and solve it by a level set method. Numerical examples are presented to verify effectiveness of the shape and topology optimization algorithms.

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Key words: Shape optimization, topology optimization, Stokes flow, hemivariational inequality, level set method.

1 Introduction

Shape and topology optimization of flows in computational fluid mechanics has many physical applications (see, e.g., monographs [1–4] and [5–10]). Typically, a shape and topology optimization problem aims to seek some design shape that minimizes a cost functional subject to certain physical constraints of partial differential equation and geometry constraint [11–13]. However, various more complex nonlinear problems, such

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as the fluid models with threshold boundary conditions, can be described by variational inequalities. The so-called threshold boundary conditions including slip and leak boundary conditions of Tresca type were introduced to the Stokes and Navier-Stokes models [14]. A possible way to express the boundary condition is to introduce appropriate nonsmooth convex functions, which leads to variational inequalities. Shape optimization for a variational inequality in Stokes fluid was considered with slip [15] and leak boundary conditions [16,17], where the existence of a solution to optimal shape design problems was studied. Furthermore, the shear stress at the optimized boundary was characterized as a subgradient of a convex functional [15]. It is worth noting that the nonsmoothness of the variational inequalities will create difficulties in numerical computation since the classical gradient type method will fail. Smoothing techniques were used to perform sensitivity analysis [15–17].

Shape optimization of elliptic variational inequalities with elliptic systems [18, 19] can have applications in, e.g., superconductors [20] and damage models [21]. For shape and topological designs of structures governed by variational inequalities, we refer to see [4, 22] and [23, 24], respectively.

Variational inequalities are mathematical problems with convex structures, including arguments of monotonicity and properties of the subdifferential of a convex function. In contrast, there are a number of nonlinear boundary value problems with nonconvex structures involving nonmonotone relations between physical quantities. In such cases, as extensions of variational inequalities, hemivariational inequalities are required for modelling. Hemivariational inequality first introduced by Panagiotopoulos [25] was closely related to the development of the concept of the generalized gradient of a locally Lipschitz continuous function [26]. The theory and applications of hemivariational inequalities have been studied in [27–32]. Numerical methods were applied to solve hemivariational inequalities for the Stokes and Navier-Stokes equations with a nonlinear slip boundary condition [33, 34].

The shape and topology optimization with a design purpose of fluids governed by hemivariational inequalities can have physical applications in, e.g., contact and friction. Shape optimization for the Stokes hemivariational inequality with slip boundary condition was first considered in [35], where existence and convergence analysis of a solution to the two-dimensional optimal shape design model problems of the Stokes hemivariational inequality were presented, while no sensitivity analysis and numerical experiments were done.

In literature, there are research works on topology optimization of incompressible fluid flows modelled with variational equalities (density method [36], level set methods [37–39], phase field methods [40, 41], etc). To the best of our view, however, there is no research work on topology optimization governed by variational and hemivariational inequalities in fluid flows.

In this paper, we consider modelling and numerical shape and topology optimization for a hemivariational inequality in Stokes fluid. After smoothing, an Uzawa algorithm is used for the mixed finite element solution of the hemivariational inequality. We consider