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A Penalty-Free or Penalty-Factor-Free DG Method with a Locally Reconstructed Curl Operator for the Maxwell Eigenproblem

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Abstract. A new discontinuous Galerkin (DG) method is proposed and analyzed for the Maxwell eigenproblem, featuring a local reconstruction of the curl operator in a discontinuous finite element space. The proposed method can be penalty-free or penalty-factor-free, depending on which discontinuous finite element space the curl operator is locally reconstructed in. The new DG method is recast into a saddle-point problem so that it can be analyzed from the Babuška-Osborn theory for the finite element approximation of the spectrum of the compact operator, and the convergence and the optimal error estimates are then obtained; the discrete eigenmodes are spurious-free and spectral-correct. We provide numerical results to illustrate the proposed method.

AMS subject classifications: 65N30

Key words: Maxwell eigenproblem, discontinuous Galerkin finite element method, local reconstruction, curl operator, convergence, error estimates.

1 Introduction

In mathematical and computational studies of electromagnetism, the Maxwell eigenproblem and its numerical methods have been interesting, and the finite element method prevails in seeking the discrete eigenmodes. In this paper, we are concerned with the discontinuous Galerkin (DG) finite element method for solving this problem, which consists

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of finding the eigenmodes (eigenvalue and eigenfunction) $(\omega^2>0,\mathbf{u}\neq 0)$ such that

$$\begin{aligned} \mathrm{curl}\,\mu^{-1}\mathrm{curl}\,\mathbf{u} &= \omega^2\varepsilon\mathbf{u}, \quad \text{in } \Omega,\\ \mathrm{div}\,\varepsilon\mathbf{u} &= 0, \quad \text{in } \Omega,\\ \mathbf{n}\times\mathbf{u} &= 0, \quad \text{on } \Gamma,\\ \int_{\Gamma_i}\mathbf{n}\cdot\varepsilon\mathbf{u} &= 0, \quad 1\le i\le n. \end{aligned} \tag{1.1}$$

In the above problem, $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a bounded, Lipschitz domain, with boundary $\Gamma=\Gamma_0\cup_{i=1}^n\Gamma_i$, where Γ_0 is the outermost connected boundary of Ω and Γ_i, $1\le i\le n$, are the other connected components of Γ. The two matrix-valued functions μ,ε describe the physical properties (magnetic permeability and electric permittivity) of the media occupying Ω. The last integral constraints account for the nontrivial topology of Ω (here Γ has disconnected components and introduces non trivial solutions for $\omega^2=0$), cf. [22].

As a finite element method, in addition to its extensive applications elsewhere, the DG method is also applied to numerically solving (1.1). To study the DG method, a first thing is to state a variational problem. A classical variational statement is to discard the divergence constraint and the integral constraints and find the eigenmodes $(\omega^2,\mathbf{u}\neq 0)\in\mathbb{R}\times H_0(\mathrm{curl},\Omega)$ such that

$$(\mu^{-1}\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{v})=\omega^2(\varepsilon\mathbf{u},\mathbf{v}), \quad \forall\mathbf{v}\in H_0(\mathrm{curl},\Omega), \tag{1.2}$$

where $H_0(\mathrm{curl},\Omega)=\{\mathbf{v}\in H(\mathrm{curl},\Omega):\mathbf{n}\times\mathbf{v}|_\Gamma=0\}$, $H(\mathrm{curl},\Omega)=\{\mathbf{v}\in(L^2(\Omega))^d:\mathrm{curl}\,\mathbf{v}\in(L^2(\Omega))^{2d-3}\}$. This problem introduces a zero eigenvalue, whose eigenfunctions span the kernel space of the curl operator. With the simplicity, however, (1.2) has been widely employed in the computation of the discrete eigenmodes. As is well-known, on the other hand, some finite element methods based on (1.2) suffer from spurious and incorrect finite element solutions. This is deeply rooted in the noncompact nature of (1.2). The so-called edge elements, which are $H(\mathrm{curl},\Omega)$-conforming, is generally suitable for spurious-free and spectral-correct discrete eigenmodes of (1.2) (cf. [27]). The examples of edge elements are the Nédélec elements on simplices [29,30]. Some conforming but non edge element methods are referred to [20] and [19].

The DG method, as an alternative, is also valid for seeking spurious-free and spectral-correct discrete eigenmodes. The DG method usually bears the flexibility in many aspects such as meshes, finite element spaces and the discretizations of the partial derivatives operators involved, etc, see a review in [2]. A common property of the DG method is the $h^{-1/2}$ penalty of the jumps, accounting for the discontinuity across the interelement boundaries and for the nonhomogeneous boundary values. A unfavorable thing in the penalty term is the demand of a penalty factor, which would be rather difficult to determine in advance and is usually problem-dependent, and it is very often manually tuned. In general, such penalty-factor needs to be large enough for the stability; in some DG methods, a large penalty-factor, which is dependent additionally on the mesh size, is introduced for reasonable convergence rates or for isolating the spurious eigenmodes from