Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2025-0007

Vol. **4**, No. 3, pp. 307-335
August 2025

On the Eigenvalue Problem for a Bulk/Surface Elliptic System

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Received 15 June 2024; Accepted 6 June 2025

Abstract. The paper addresses the doubly elliptic eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ -\Delta_\Gamma u + \partial_\nu u = \lambda u & \text{on } \Gamma_1, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with a C^1 boundary $\Gamma = \Gamma_0 \cup \Gamma_1, \Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_1 being nonempty and relatively open on Γ. Moreover, $\mathcal{H}^{N-1}(\overline{\Gamma}_0 \cap \overline{\Gamma}_1) = 0$ and $\mathcal{H}^{N-1}(\Gamma_0) > 0$. We prove that $L^2(\Omega) \times L^2(\Gamma_1)$ admits a Hilbert basis constituted by eigenfunctions and we describe the behavior of the eigenvalues. Moreover, when Γ is at least C^2 and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, we give several qualitative properties of the eigenfunctions.

AMS subject classifications: 35J05, 35J20, 35J25, 35J57, 35L05, 35L10
Key words: Bulk/surface, elliptic system, eigenvalue problem, oscillation modes, standing solutions, hyperbolic dynamic boundary conditions, wave equation.

1 Introduction and main results

1.1 Presentation of the problem and literature overview

We deal with the doubly elliptic eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ -\Delta_\Gamma u + \partial_\nu u = \lambda u & \text{on } \Gamma_1, \end{cases} \tag{1.1}$$

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where Ω is a bounded open subset of $\mathbb{R}^N$ $(N\geq 2)$ with C^r boundary Γ (see [21]), with $r=1,2,\ldots,\infty$. Hence, when nothing is said, $r=1$. We also assume that $\Gamma=\Gamma_0\cup\Gamma_1$, $\Gamma_0\cap\Gamma_1=\emptyset$, Γ_1 being nonempty and relatively open on Γ (or equivalently $\overline{\Gamma_0}=\Gamma_0$). Denoting by $\mathcal{H}^{N-1}$ the Hausdorff measure, we also assume that $\mathcal{H}^{N-1}(\overline{\Gamma}_0\cap\overline{\Gamma}_1)=0$ and $\mathcal{H}^{N-1}(\Gamma_0)>0$. These properties of Ω,Γ_0 and Γ_1 will be assumed, without further comments, throughout the paper.

In problem (1.1) λ is a real or complex parameter and we respectively denote by Δ and Δ_Γ the Laplace operator in Ω and the Laplace-Beltrami operator on Γ, while ν stands for the outward unit normal to Ω. We shall look for eigenvalues and eigenfunctions of problem (1.1), that is for values of λ for which (1.1) has a nontrivial (real or complex-valued) solution, i.e. an eigenfunction.

Problem (1.1) has been studied (as the particular case $K=0,\alpha=1$ and $\gamma=\omega$ in problem (1.2)) in [27], when $\Gamma_0=\emptyset$ and λ,u are real. The study in [27] is motivated by several papers on the Allen-Cahn equation subject to a dynamic boundary condition, see [7, 9, 40]. Indeed, finding a Hilbert basis of eigenfunctions allows to look for solutions of the evolution problem by using a Faedo-Galerkin scheme.

We remark that, by introducing an (inessential) positive parameter κ in front of the Laplacian in (1.1), and formally taking the limit as $\kappa\to\infty$, one gets from (1.1) the Wentzell eigenvalue problem, studied in [11,16], which is then related to (1.1).

As to the author's knowledge problem (1.1) in the case $\Gamma_0\neq\emptyset$ has not yet been considered in the mathematical literature. The motivation for studying it originates from an evolution problem different than the one mentioned above. Indeed, it originates from the wave equation with hyperbolic boundary conditions. It is the evolutionary boundary value problem

$$\begin{cases} w_{tt}-\Delta w=0 & \text{in } \mathbb{R}\times\Omega,\\ w=0 & \text{on } \mathbb{R}\times\Gamma_0,\\ w_{tt}-\Delta_\Gamma w+\partial_\nu w=0 & \text{on } \mathbb{R}\times\Gamma_1, \end{cases} \tag{1.2}$$

where Ω,Γ_0 and Γ_1 are as above, $w=w(t,x),t\in\mathbb{R},x\in\overline{\Omega},\Delta=\Delta_x$ and Δ_Γ denote the Laplacian and Laplace-Beltrami operators with respect to the space variable.

One easily see that solutions of (1.2) enjoy energy conservation, once a properly defined energy function is introduced. So, while one cannot expect decay of solutions, it is of interest to look for standing wave solutions of (1.2). They are solutions of the form

$$w(t,x)=e^{i\omega t}u(x),\quad \omega\in\mathbb{R}\setminus\{0\}, \tag{1.3}$$

where u is nontrivial and real-valued. The function w defined in (1.3) solves, at least formally, problem (1.2) if and only if u solves problem (1.1) with $\lambda=\omega^2>0$.