Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2025-0008

Vol. **4**, No. 3, pp. 336-346
August 2025

Uncertainty and Strichartz Type Inequalities for Fractional Hajłasz-Sobolev Spaces

Joaquim Martín[1,*] and Walter A. Ortiz[2]

[1] *Department of Mathematics - Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain.*
[2] *School of Engineering, Science and Technology - Universidad Internacional de Valencia, 46002 Valencia, Spain.*

Received 1 October 2024; Accepted 8 June 2025

Abstract. In this paper we obtain uncertainty and Strichartz type inequalities for fractional Hajłasz-Sobolev spaces in the metric setting.

AMS subject classifications: 46E35, 46E30, 30L99

Key words: Metric measure space, rearrangement invariant spaces, uncertainty inequality, Besov spaces.

---

1 Introduction

Consider a metric measure space (Ω,d,μ), where μ is a Borel measure on (Ω,d) such that $0<\mu(B)<\infty$ for every ball B in Ω. We will always assume $\mu(\Omega)=\infty$ and $\mu(\{x\})=0$ for all $x\in\Omega$. Let $s>0$ and let X be a rearrangement invariant (r.i.) space on Ω (see Section 2 below), given $f\in X$, a function $g\in X$ is said to be an s-gradient of f if the following inequality holds:

$$|f(x)-f(y)|\le d(x,y)^s\left(g(x)+g(y)\right)\quad \mu\text{-a.e.}\quad x,y\in\Omega.$$

We denote by $D^s(f)$ the collection of all s-gradients of f.

---

*Corresponding author. *Email addresses:* `walterandres.ortiz@professor.universidadviu.com` (W. A. Ortiz), `Joaquin.Martin@uab.cat` (J. Martín)

The fractional Hajłasz-Sobolev space $M^{s,X}(\Omega)$ consists of all functions $f\in X$ for which the norm

$$\|f\|_{M^{s,X}(\Omega)}=\|f\|_X+\inf_{g\in D^s(u)}\|g\|_X$$

is finite (see [17]).

When $X=L^p$, the spaces $M^{1,L^p}(\Omega)$ $(1\le p<\infty)$ were first introduced by Hajłasz (see [4,5]) in order to obtain extensions of the classical theory of Sobolev spaces to the setting of metric spaces (for $p>1, M^{1,L^p}(\mathbb{R}^n)=W^{1,p}(\mathbb{R}^n)$ (see [4]), whereas for $p=1, M^{1,1}(\mathbb{R}^n)$ coincides with the Hardy-Sobolev space $H^{1,1}(\mathbb{R}^n)$ (see [13, Theorem 1])). Hajłasz-Sobolev spaces play an important role in the area of analysis known as analysis on metric spaces, and a large number of papers have focused on them (see, for example, [1,7–10], and the references quoted therein). When the measure μ is doubling[†], spaces $M^{1,X}(\Omega)$ have been considered in some particular cases, for example, Hajłasz-Lorentz-Sobolev spaces $M^{1,L^{p,q}}(\Omega)$ (see [12]) and Musielak-Orlicz-Hajłasz-Sobolev spaces $M^{1,L^\Phi}(\Omega)$, where L^Φ is an Orlicz space (see [19]). Also in the doubling case, fractional spaces $M^{s,L^p}(\Omega)$ were introduced and studied in [6,11,21] (if $0<s<1$, then $M^{s,L^p}(\mathbb{R}^n)=B^s_{p,\infty}(\mathbb{R}^n)$ (see [21])).

The purpose of this paper is to obtain uncertainty and Strichartz type inequalities for fractional Hajłasz-Sobolev spaces. The theory of isoperimetric weights and generalised uncertainty inequalities in metric measure spaces developed by Martín and Milman [16] has been a motivation and a model for the research presented here.

The structure of the paper is as follows, in Section 2, we introduce the notation and standard assumptions used in the paper, and in Section 3, we obtain uncertainty and Strichartz type inequalities for fractional Hajłasz-Sobolev spaces and give some examples.

Throughout the paper, we shall write $f\preceq g$ instead of $f\le Cg$ for some constant $C>0$.

2 Notations and preliminaries

In this section, we establish some further notation and background information, and we give more details about the metric spaces and r.i. spaces we will be working with.

Let (Ω,d) be a metric space. As usual, a ball B in Ω with centre x and radius $r>0$ is a set $B=B(x,r):=\{y\in\Omega; d(x,y)<r\}$. Throughout the paper by a metric

[†] μ is said to be doubling provided there exists a constant $C>0$ such that $\mu(2B)\le C\mu(B)$ for all balls $B\subset\Omega$.