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Homogenization of an Elliptic System Involving Non-Local and Equi-Valued Interface Conditions

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Abstract. In this paper, we analyze the effective behaviour of the solution of an elliptic problem in a two-phase composite material with non-standard imperfect contact conditions between its constituents. More specifically, we consider on the interface an equi-valued surface condition and a non-local flux condition involving a scaling parameter α. We perform a homogenization procedure by using the periodic unfolding technique. As a result, we obtain two different effective models, depending on the scaling parameter α. More precisely, in the case $\alpha > -1$, we are led to a standard Dirichlet problem for an elliptic equation, while in the case $\alpha = -1$, we get a bidomain system, consisting in the coupling of an elliptic equation with an algebraic one.

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1 Introduction

In this paper, we study the homogenization of a stationary heat diffusion problem in a two-phase composite material with imperfect contact conditions between its

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constituents. We assume that the material has a periodic microstructure of characteristic length $\varepsilon > 0$, where ε is a small parameter which in the homogenization procedure will be let tend to 0.

More precisely, inspired by the paper of Geng (see [17, Table 1, Case 3]), on the interface separating the two phases, we consider the equi-valued surface condition (2.5) and the non-local one (2.7). In particular, the last condition contains a scaling parameter α, which is related to the speed of the interfacial heat exchange, and we perform the homogenization for all the admissible values of such a parameter, i.e. $\alpha \geq -1$.

The homogenization process is carried out by applying the periodic unfolding technique described in Section 3 (see, for instance, [12, 15]). As a result of such a procedure, we obtain two different effective models, depending on the scaling parameter α. More precisely, in the case $\alpha > -1$, we are led to a standard Dirichlet problem for an elliptic equation, while in the case $\alpha = -1$, we get a bidomain system, consisting in the coupling of an elliptic equation with an algebraic one. Despite the fact that all the scalings $\alpha > -1$ are governed by the same limit problem (4.31), the proof of the different cases $\alpha > 1, \alpha = 1$ and $-1 < \alpha < 1$ needs different tools; in particular, different test functions, whose construction is not trivial, are required. We also point out that, in contrast with the standard form of the limit model (at least in the case $\alpha > -1$), the proof is not a routine one, due to the non-standard functional setting.

Finally, we stress that, in our case, the cell problem, commonly involved in the homogenization procedure, is not the expected one. This is due to condition (4.3), but mainly to the fact that, in contrast with the typical cell problem structure, usually reproducing the microscopic one in the periodic setting, here the jumps of the microscopic heat potential present in condition (2.7) are lost by our cell functions, which are continuous across the interface.

This paper can be set in the same research area of a lot of mathematical contributions published in the last decades (see, for instance, [2, 3, 5–9, 16, 20]). However, up to our knowledge, the results presented here are new and could deserve some interest for possible engineering applications, recently focused on models with non-standard imperfect contact conditions (see, for instance, [18, 19] and the references therein).

The paper is organized as follows. In Section 2, we state our microscopical problem and we prove the main energy estimates satisfied by its solution. In Section 3, we recall the definition of the unfolding operators used throughout this paper and their main compactness properties. The section also contains the specific compactness results for the unique solution of our microscopic variational problem. Finally, Section 4 is devoted to the statement and the proof of our main