Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2025-0010

Vol. **4**, No. 3, pp. 368-391
August 2025

Multi-Level Structured Deformations: Relaxation via a Global Method Approach

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Received 18 December 2024; Accepted 6 June 2025

Abstract. We present some relaxation and integral representation results for energy functionals in the setting of structured deformations, with special emphasis given to the case of multi-level structured deformations. In particular, we present an integral representation result for an abstract class of variational functionals in this framework via a global method for relaxation and identify, under quite general assumptions, the corresponding relaxed energy densities through the study of a related local Dirichlet-type problem.
Some applications to specific relaxation problems are also mentioned, showing that our global method approach recovers some previously established results.

AMS subject classifications: 49J45, 46E30, 74A60, 74M99, 74B20

Key words: Global method for relaxation, hierarchical system of structured deformations, multi-scale geometry, disarrangements, integral representation.

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1 Introduction

The purpose of this work is to give an overview of some relaxation and integral representation results in the context of structured deformations. The main focus is to present, and give an improvement, of some recent findings, obtained in [8], where we established a global method for relaxation that is applicable in the context of multi-level structured deformations. We also provide several applications that show that our approach covers, and in some cases actually improves, in a unified way, various results available in the literature in the setting of (first-order) structured deformations.

The aim of the theory of (first-order) structured deformations (see [12]) is to describe the effects, at the macroscopic level, of both smooth and non-smooth geometrical changes that a body may undergo and that occur at one sub-macroscopic level. A further step in this theory was undertaken in [14], where the notion of a hierarchical system of structured deformations was introduced in order to include the effects, at the macro-level, of geometrical changes occurring at more than one sub-macroscopic level. We refer to Sections 2 and 4 for more details.

Since the seminal papers of Del Piero and Owen [12], where structured deformations were originally introduced, and of Choksi and Fonseca [11], where the variational formulation for (first-order) structured deformations in the space of functions with bounded variation (SBV) setting was first addressed, the theory has known many generalisations and extensions.

For example, the notion of second-order structured deformations was introduced by Owen and Paroni [20], in order to describe curvature and bending effects associated with jumps in gradients. Lower semi-continuity, relaxation and integral representation results in several functional spaces, such as the space of functions with bounded Hessian (BH), the space of special functions with bounded Hessian (SBH) and the space of functions with bounded variations such that the absolutely continuous part of the gradient is special function with bounded variation (SBV^2), and with respect to different topologies, have been obtained in [4,5,15,16,21,22], for example.

Further contributions include applications to dimension reduction problems as in [10] and to homogenisation [2]. A recent article by Krömer *et al.* [17] extends the notion of structured deformations to so-called measure structured deformations. We also refer to the survey by Matias *et al.* [18] where an overview of many of these topics may be found.

The global method for relaxation that we proposed in [8] allows us to provide an integral representation for a class of functionals defined in the set of $(L+1)$-level (first-order) structured deformations, $(g,G_1,\dots,G_L)\in HSD_L^p(\Omega)$, where, for