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Crossing Thresholds in High Contrast Boundary Homogenization Problems: A Review

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Abstract. In this paper we provide a review of the state of the art in two topics of the homogenization theory: boundary homogenization for rapidly alternating boundary conditions on grill-type walls and homogenization of linear or nonlinear Robin boundary conditions in perforate media. High contrasts on the Robin boundary conditions are introduced. These contrasts are represented by new parameters and, along with the period and the sizes of "grill-crossing points"/perforations, play an important role in determining "thresholds" marking changes in the homogenized boundary conditions, either with "averaged terms" or "strange terms" or, somehow, "extreme" asymptotic behaviors for the solutions. Taking into account the advances of the last years in the topics, we also overview vector problems (cf. Winkler-Robin boundary conditions) and models in perforated media of interest in material sciences among others.

AMS subject classifications: 35B27, 74Q20, 35J65, 35J88, 35J05, 35P05, 35C20

Key words: Boundary homogenization, high contrasts, nonlinear boundary conditions, Robin conditions, Winkler conditions, strange term, critical relations.

1 Introduction

During the decade from the second half of the 1970s to the first half of the 1980s, many works devoted to the so-called "the crushed ice problem", "the cloud of ice", "the Neumann's strainer", "the fakir's bed of nails", "the Dirichlet's archi-

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pelago", among others, appeared in the literature of applied mathematics; cf., e.g. [7,26,110,114].

In particular, [26,77,110], consider the homogenization for the Laplace operator in a perforated domain of $\mathbb{R}^n, n\geq 2$, with a Dirichlet boundary condition obtaining what they called a "strange term", namely a new term of order $\mathcal{O}(1)$ in the partial differential equation which is obtained in some way from the average of the heterogeneous media ("density of holes" and the boundary conditions); it contains a "capacity term". Similarly, the Neumann problem has been considered, cf. [30]. These problems have the common fact that the "size" of the cavities r_ε can be much smaller than the characteristic period of the perforation ε and depending on the different relations between these parameters, different homogenized boundary conditions appear, as $\varepsilon\to 0$. For the Dirichlet Laplacian, "extreme cases" arise when the size of the perforations is too large and the solution asymptotically becomes zero or the perforations are so small that the solution asymptotically ignores them. The intermediate case is provided by the so-called "critical size" of the perforations giving rise to the above mentioned strange term in the equation. Different methods are used in these frameworks, let us mention energy methods, oscillating functions, multiscale methods, G-convergence of operators, Γ-convergence, unfolding methods, etc. Additionally, extensions to other operators such as those involved with heterogeneous media, the p-Laplacian operator, the elasticity operator or Stokes equations have been considered early in the literature, cf. [2,7,78,101] among others, and references therein. Also, the homogenization for unilateral constraints ("obstacles problems") were considered, cf. [7] and references therein.

At the same time as this spreading in perforated domains over the entire volume, and in the framework of "critical sizes" and "extreme cases", it develops the boundary homogenization problems with the perforations placed along a hyperplane inside the domain. These problems somehow are variants of the above-mentioned Neumann strainer and Dirichlet archipelago in [114], with "plane holes" (domains in $\mathbb{R}^{n-1}$), which may represent complementary studies for fluids throughout perforated walls, cf. [112,122], or variants of former formulations, cf. [87] and references therein. See also [41,42] for related industrial problems. The general limit behavior is as above mentioned, but, with the averaged or strange terms arising on the boundary conditions or on interface conditions, cf. [2,7,13,15,26,33,34,84,92,108] for different operators and, e.g. [6,8,20,21,75,83] for different variants of the problem, different results and more recent references on the subject.

Besides, the homogenization of Robin boundary conditions, also known as the third type boundary conditions, on perforated media attracted the interest