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Inviscid Limit of the Navier-Stokes-Korteweg Equations under the Weak Kolmogorov Hypothesis in $\mathbb{R}^3$

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Dedicated to Professor Gui-Qiang Chen on the occasion of his 60th birthday

Abstract. This paper establishes the weak convergence of global solutions for the Navier-Stokes-Korteweg equations under the weak Kolmogorov hypothesis in the three-dimensional periodic domain. Specifically, the weak Kolmogorov hypothesis offers uniform bounds for weak solutions, ensuring their weak stability under vanishing viscosity. With compactness arguments, we show that the solutions of the Navier-Stokes-Korteweg equations converge to a global weak solution of the Euler-Korteweg equations.

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Key words: Inviscid limit, Kolmogorov hypothesis, Navier-Stokes-Korteweg equations, Euler-Korteweg equations, compactness.

1 Introduction

In this article, we are concerned with the vanishing viscosity limit of the following three-dimensional Navier-Stokes-Korteweg equations:

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$$\begin{cases} \rho_t+\operatorname{div}(\rho\mathbf{u})=0, & \text{(1.1a)} \\ (\rho\mathbf{u})_t+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla\rho^\gamma-\operatorname{div}(\sqrt{\mu\rho}\mathbb{S})=\kappa\rho\nabla\Delta\rho+\rho\mathbf{f}, & \text{(1.1b)} \end{cases}$$

where ρ denotes the fluid density, $\mathbf{u}$ represents the velocity vector field, $\mu>0$ stands for the viscosity coefficient, and $\mathbf{f}=\mathbf{f}(\mathbf{x},t)=(f_1,f_2,f_3)(\mathbf{x},t)$ denotes an externally applied force. As in [8] we write $\sqrt{\mu\rho}\mathbb{S}=\mu\rho\mathbb{D}\mathbf{u}$, where

$$\mathbb{D}\mathbf{u}:=\frac{\nabla\mathbf{u}+\nabla^\top\mathbf{u}}{\mathbf{2}}$$

is the strain tensor. The Korteweg stress term, denoted by σ_κ, is expressed as follows:

$$\sigma_\kappa=\kappa\nabla\Delta\rho,$$

where κ is a coefficient representing the strength of capillarity or surface tension effects. For more discussion on the Korteweg stress term, we refer the readers to [4–6]. It is a term added to the Navier-Stokes equations to account for capillary effects and surface tension in fluid flow. It is particularly relevant in situations where fluid interfaces are present, such as in the behavior of fluids near solid surfaces or at the interface between immiscible fluids. The inclusion of the Korteweg term in the Navier-Stokes equations introduces nonlinear effects that can significantly influence the behavior of fluid flow, especially in situations where surface tension plays a significant role. For general ideal barotropic fluids, the pressure law is given by $P=a\rho^\gamma$,where $\gamma>1$, is the adiabatic exponent and $a>0$ is a constant, and the internal energy is

$$e=\frac{P}{(\gamma-1)\rho}=\frac{c^2}{\gamma-1},$$

where c is the sonic speed. In particular, the case of $\gamma=2$, the two-dimensional system becomes the shallow water equations, see [14]. In this paper, we will consider the general Navier-Stokes-Korteweg equations in three dimensional space for any $\gamma>1$ and take $a=1$ without loss of generality.

The objective of this paper is to investigate the limit as the viscosity coefficient μ tends to zero in the Navier-Stokes-Korteweg equations (1.1) within the periodic domain $\mathbb{T}^3\subset\mathbb{R}^3$. We focus on the initial value problem of (1.1) subject to the initial condition

$$(\rho,\rho\mathbf{u})|_{t=0}=(\rho_0,\mathbf{m}_0)(\mathbf{x}),\quad \mathbf{x}\in\mathbb{T}^3 \tag{1.2}$$

under Kolmogorov's well-established hypothesis [11, 12]. Specifically, we examine a modified version of Assumption (KH), denoted as Assumption (KHw),