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Asymptotic Analysis of Steady Viscous Shocks in a 1-D Finite Nozzle in the Small Viscosity Limit

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Dedicated to Professor Gui-Qiang Chen on the occasion of his 60th birthday

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Abstract. In this paper we show that, as the viscosity is properly small, there exists a viscous transonic shock solution for the steady 1-D Navier-Stokes system with prescribed pressure at the exit, and it converges to a transonic shock solution to the 1-D steady Euler system as the viscosity goes to zero. Moreover, the position of the shock front is also derived. The key step is to reduce the pressure condition at the exit into a nonlinear boundary condition on the velocity, such that the boundary value problem for the Navier-Stokes system can be reformulated as a boundary value problem for an ODE with an unknown parameter.

AMS subject classifications: 35A02, 35L65, 35L67, 35Q31, 76L05, 76N10, 76N17

Key words: Asymptotic analysis, steady 1-D viscous shocks, 1-D finite nozzle, small viscous limits, Euler system, Navier-Stokes system.

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1 Introduction

In this paper, we are going to investigate the asymptotic behavior of one-dimensional (1-D) steady viscous shock solutions to the Navier-Stokes (NS) system in a finite nozzle with prescribed pressure at the exit as the viscosity goes to zero. In particular, we are concerned with the case that the viscous solutions converge to a shock solution of 1-D steady Euler system. Moreover, we can determine the shock position via the viscous limit.

Let the finite nozzle be bounded in an interval

$$\mathcal{N} := \{x \in \mathbb{R} : 0 < x < 1\}.$$

The viscous flow in the nozzle is governed by the following 1-D steady Navier-Stokes system:

$$\partial_x(\rho_\varepsilon u_\varepsilon) = 0, \tag{1.1}$$

$$\partial_x\left(\rho_\varepsilon u_\varepsilon^2 + p_\varepsilon\right) = \varepsilon \partial_{xx} u_\varepsilon, \tag{1.2}$$

$$\partial_x(\rho_\varepsilon u_\varepsilon \Phi_\varepsilon) = \varepsilon \partial_x(u_\varepsilon \partial_x u_\varepsilon), \tag{1.3}$$

where $\varepsilon > 0$ is the viscosity coefficient, u_ε is the velocity, $(p_\varepsilon, \rho_\varepsilon)$ represents the pressure and the density, and $\Phi_\varepsilon = u_\varepsilon^2/2 + i_\varepsilon$ with $i_\varepsilon = e_\varepsilon + p_\varepsilon/\rho_\varepsilon$ the enthalpy and e_ε the internal energy. The fluid in the nozzle is assumed to be a polytropic gas which satisfies the following state equation with $\gamma > 1$ the adiabatic exponent:

$$e_\varepsilon = \frac{1}{\gamma - 1} \frac{p_\varepsilon}{\rho_\varepsilon}. \tag{1.4}$$

Then the sonic speed of the flow is given by

$$c_\varepsilon^2 = \frac{\gamma p_\varepsilon}{\rho_\varepsilon}, \tag{1.5}$$

and let $M_\varepsilon := |u_\varepsilon|/c_\varepsilon$ be the Mach number. The flow is called supersonic as $|u_\varepsilon| > c_\varepsilon$, and subsonic as $|u_\varepsilon| < c_\varepsilon$.

Recently, Fang and Zhao [16] have studied the asymptotic behavior of the viscous shock solutions to the steady 1-D Navier-Stokes system as the viscosity vanishes and observed that they converge to a shock solution of the steady Euler system. It is interesting that, although there are infinite many shock solutions to the Euler system for the inviscid flow, they showed that only one of them could be the limit of the viscous solutions. This phenomenon is further confirmed in [13] for the case that the viscosity coefficients depend on the temperature. Both in [13]