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SBV-like Regularity of Entropy Solutions for a Scalar Balance Law

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Dedicated to Professor Gui-Qiang Chen on the occasion of his 60th birthday

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Abstract. In this note, we demonstrate that solutions to scalar balance laws, in one space dimension, which exhibit bounded variation, must be functions of special bounded variation (SBV). This case study illustrates the strategy applied in [Ancona et al., preprint, University of Padova, 2024] to systems of balance laws, extending the methodologies developed in pioneering previous works by several authors. While for a single balance law a more general work is already available, generalizing the first breakthrough related to a conservation law, the case of $1D$ systems presents new behaviors that require a different strategy. This is why in this note we make the effort to introduce the notation and tools that are required for the case of more equations. When the flux presents linear degeneracies, it is known that entropy solutions can really present nasty fractal Cantor-like behaviours, although $f'(u)$ is still SBV: We thus discuss SBV-like regularity, as SBV-regularity fails.

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1 Introduction

In this note we discuss the SBV-regularity for a scalar balance law in one space dimension as a case study in order to explain the strategy that we apply in [3] to systems of balance laws, generalizing [5–7]. We stress that for a single balance law the more general work [18] is already available, generalizing the breakthrough [1] related to a conservation law, nevertheless the case of $1D$ systems presents new behaviors that require a different strategy. This is why in this note we make the effort to use the notation and tools that are required for the case of more equations, although one equation is simpler.

While in the note [9] the heuristics of the new approach was presented for Burgers' equation, with no source, in the present note we will also provide an insight of the tools introduced towards the regularity estimate.

Consider the Cauchy problem for a single balance law in one space dimension

$$\partial_{\mathrm{t}} u(t,x)+\partial_{\mathrm{x}} f\big(u(t,x)\big)=g(t,x,u) \quad \text{together with} \quad \overline{u}=u(0,\cdot)\in \mathrm{BV}(\mathbb{R}). \tag{1.1}$$

We assume that the source term satisfies the following assumption:

(G) The function $g:\mathbb{R}^3\to\mathbb{R}$ is continuous in t and Lipschitz continuous with respect to x and u, uniformly in t; moreover, suppose there exists a function $\alpha\in L^1(\mathbb{R})$ such that $|g_x(t,x,u)|\le\alpha(x)$ for any t,u.

Definition 1.1 (Cantor Part of the Derivative of a BV-function of One Variable)**.** *Suppose $v:(a,b)\to\mathbb{R}$ has bounded variation, thus D_xv can be identified with a measure. We call Cantor part of D_xv, and we denote it by $D_x^{\mathrm{Cantor}}v$, the continuous part of the measure D_xv which is not absolutely continuous in the Lebesgue 1-dimensional measure: We write*

$$D_xv=D_x^{\mathrm{a.c.}}v+D_x^{\mathrm{Cantor}}v+D_x^{\mathrm{jump}}v,$$

where $D_x^{\mathrm{a.c.}}v$ is absolutely continuous and $D_x^{\mathrm{jump}}v$ is purely atomic. If $D_x^{\mathrm{Cantor}}v=0$ we say that v is a special function of bounded variation and we denote $v\in\mathrm{SBV}((a,b))$.

As a convention all through the note, when we restrict a function of bounded variation u to some time t we think that $u(t)$ is the representative continuous from the right, in time. As well, $D_xu(t)$ will be w^*-continuous from the right.

Theorem 1.1. *Let $\mathfrak{c}>0$. Consider the entropy solution $u:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ to the Cauchy problem for the balance law* (1.1). *Suppose that $f\in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ satisfies $f'(z+h)-f'(z)\ge\mathfrak{c}h$ for $h>0$, and that g satisfies Assumption* **(G)**.

Then $x\mapsto u(t,x)$ is a special function of bounded variation for $t\notin S$, with S at most countable.