Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2024-0004

Vol. **4**, No. 1, pp. 1-16
February 2025

On the Sum of Operators of p-Laplacian Types

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Received 27 August 2023; Accepted 2 December 2023

Abstract. The goal of this paper is to study operators sum of p-Laplacian type operators. We address the problems of existence and uniqueness of solutions, this last point leading to some challenging issues in the case of quasilinear combinations of such p-Laplacians.

AMS subject classifications: 35A15, 35J60, 35J66, 35J92

Key words: p-Laplacian, nonlinear elliptic operators, existence, uniqueness.

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1 Introduction and notation

We will denote by Ω a bounded open subset of $\mathbb{R}^n, n\geq 1$. Let us consider $p_1,p_2,\ldots,$ p_N real numbers such that

$$1<p_1<p_2<\ldots<p_N,$$

and $a_i(x,u), i=1,\ldots,N$, Carathéodory functions, i.e. such that for every i,

$$\begin{aligned} x &\to a_i(x,u) \quad \text{is measurable}, \quad \forall u\in\mathbb{R},\\ u &\to a_i(x,u) \quad \text{is continuous a.e. } x\in\Omega. \end{aligned}$$

We will suppose that for some positive constants λ,Λ,

$$\begin{aligned} &0\leq a_i(x,u)\leq\Lambda, \quad \forall i=1,\ldots,N-1,\\ &\lambda\leq a_N(x,u)\leq\Lambda, \quad \forall u\in\mathbb{R} \quad \text{a.e. } x\in\Omega. \end{aligned} \tag{1.1}$$

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We would like to consider problems of the following type:

$$\begin{cases} u\in W_0^{1,p_N}(\Omega), \\ -\nabla\cdot\left(\displaystyle\sum_{i=1}^N a_i(x,u)|\nabla u|^{p_i-2}\nabla u\right)=f \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

or under the weak form

$$\begin{cases} u\in W_0^{1,p_N}(\Omega), \\ \displaystyle\int_\Omega \sum_{i=1}^N a_i(x,u)|\nabla u|^{p_i-2}\nabla u\cdot\nabla v dx=\langle f,v\rangle, \quad \forall v\in W_0^{1,p_N}(\Omega), \end{cases} \tag{1.3}$$

where $W_0^{1,p}(\Omega)$ denotes the usual Sobolev space of functions in $L^p(\Omega)$ with derivatives in $L^p(\Omega)$, vanishing on the boundary of $\Omega, f\in W^{-1,p'_N}(\Omega)$ the dual space of $W_0^{1,p_N}(\Omega)$ (cf. [5]). Recall that for $p\in\mathbb{R}, p>1, p'$ denotes the conjugate of p given by $p'=p/(p-1)$.

We suppose that $W_0^{1,p}(\Omega)$-spaces are equipped with the norm

$$\|\nabla v\|_p=\left(\int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}},$$

and their duals $W^{-1,p'}(\Omega)$ with the strong dual norm defined as

$$|f|_*=\sup_{v\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{|\langle f,v\rangle|}{\|\nabla v\|_p}.$$

Such operators appeared some decades ago in particular as Euler equation of problems of calculus of variations (cf. [7], [8], [10]), the idea being to consider energy functionals presenting at the same time different growth and to analyse the regularity of the possible minimisers (see [9], which contains many interesting references, and also [6]). Later (cf. [4], [11]) problems of this type were supposed to model situations where different phases coexist, two in general, leading to the notion of (p,q)-Laplacian. Of course here we consider the sum of several pseudo p-Laplacians and the Eq. (1.3) is not the Euler equation of some energy except perhaps in the case when the a_i's are constant. We do not pretend either having in mind applications. We are more guided by the challenges offered by this kind of problems when existence and uniqueness of solution are concerned.

In the next section we develop a theory of existence of solution based on the theory of monotone operators. The subsequent part addresses different issues of