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# Upper Bounds for Korn's Constants in General Domains

Cristinel Mardare[1,*] and Thai Ha Nguyen[2]

[1] *Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris, France.*
[2] *Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, SAR, P.R. China.*

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**Abstract.** We estimate the constants appearing in Korn inequalities in terms of the norm of a linear and continuous inverse to the divergence operator defined on a same domain, and of a few scalar parameters modeling the shape of the domain.

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**Key words**: Korn inequalities, shells, asymptotic analysis.

## 1 Introduction

It is well-known that, given any domain $\Omega \subset \mathbb{R}^d, d \geq 2$, and any non-empty relatively open subset $\Gamma_0$ of the boundary of $\Omega$, there exist constants $C_1 = C_1(\Omega)$, $C_2 = C_2(\Omega,\Gamma_0), C_3 = C_3(\Omega)$ and $C_4 = C_4(\Omega)$ such that

$$\inf_{\boldsymbol{r}\in \mathrm{Rig}(\Omega)} \|\boldsymbol{u}-\boldsymbol{r}\|_{H^1(\Omega)} \leq C_1 \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}, \qquad \forall \boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d), \tag{1.1}$$

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C_2 \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}, \qquad \forall \boldsymbol{u}\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d), \tag{1.2}$$

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C_3 \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)} + C_4\|\boldsymbol{u}\|_{L^2(\Omega)}, \quad \forall \boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d). \tag{1.3}$$

*Corresponding author. *Email addresses:* tthnguyen3-c@my.cityu.edu.hk (T. H. Nguyen), cristinel.mardare@sorbonne-universite.fr (C. Mardare)

Inequalities (1.2) and (1.3) constitute respectively the first and second Korn inequalities, according to most textbooks, especially in the theory of elasticity. Various proofs have been given to these inequalities, see, e.g. Duvaut and Lions [6], Fichera [7], Friedrichs [8], Gobert [9], Hlaváček [10], Hlaváček and Nečas [11], Miyoshi [16], Mosolov and Myasnikov [17], Nitsche [18], Temam [19].

The dependence of these constants on the domain $\Omega$, and on $\Gamma_0$ for the second constant, is however not well known, save for an upper bound of order of $(r/R)^d$ for domains $\Omega$ contained in a ball with radius $R$ and star-shaped with respect to a ball of radius $r$, and for some sharper upper bounds for particular domains $\Omega$, see, e.g. Horgan [12, 13], Kondratev and Oleinik [14], Ciarlet *et al.* [5]. These sharper bounds are often needed in solid and fluid mechanics in domains depending on a small parameter, where the magnitude of Korn's constant with respect to this parameter is essential in justifying dimensionally reduced models by convergence theorems when the parameter goes to zero or to infinity.

The objective of this paper is to give new proofs to the three Korn inequalities mentioned above, based on a new approach that has the advantage of yielding constants that depend explicitly on several parameters associated with the domain $\Omega$. More specifically, this new approach yields constants $C_1,\ldots,C_4$ that depend explicitly on (an upper bound $K(\Omega)$ of) the norm of a linear and continuous inverse to the divergence operator in the domain $\Omega$ (see Lemma 3.1), beside the constants appearing in Poincaré's, Poincaré-Wirtinger's, and trace, inequalities in Sobolev spaces.

The paper is organised as follows. Section 2 specifies the notation and definitions used in all ensuing sections. Section 3 estimates the constant $C_1$ appearing in Korn's inequality (1.1). The key result is the first inequality of Theorem 3.1, which provides a first estimate of the constant $C_1$ (see Corollary 3.1). Then the results of Theorem 3.1 and Corollary 3.1 are generalized, and improved, in Theorem 3.2. Section 4 estimates the constant $C_2$ appearing in Korn's inequality (1.2). The main result is Theorem 4.1, which is proved by combining Theorem 3.1 with Poincaré's inequality, trace inequality, and an inequality about the eigenvalues of symmetric matrices (Lemma 4.1). Section 5 estimates the constants $C_3$ and $C_4$ appearing in Korn's inequality (1.3). The main result is Theorem 5.3, which generalizes the previous Theorem 5.2 (at the expense of considerably more difficult and at places technical proof), itself a simpler generalization of Theorem 5.1. The proofs of all three theorems rely on the inequalities established in Theorem 3.2, combined with a method based on Fubini's theorem to estimate the norm of the anti-symmetric matrix appearing in these inequalities. Finally, Section 6 summarises all the estimates obtained in this paper for the constants $C_1, C_2, C_3$ and $C_4$, with all the details necessary to apply them in future works without having to read the entire paper.