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The Biot Stress-Right Stretch Relation for the Compressible Neo-Hooke-Ciarlet-Geymonat Model and Rivlin's Cube Problem

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Abstract. The aim of the paper is to recall the importance of the study of invertibility and monotonicity of stress-strain relations for investigating the non-uniqueness and bifurcation of homogeneous solutions of the equilibrium problem of a hyperelastic cube subjected to equiaxial tensile forces. In other words, we reconsider a remarkable possibility in this nonlinear scenario: Does symmetric loading lead only to symmetric deformations or also to asymmetric deformations? If so, what can we say about monotonicity for these homogeneous solutions, a property which is less restrictive than the energetic stability criteria of homogeneous solutions for Rivlin's cube problem. For the Neo-Hooke type materials we establish what properties the volumetric function h depending on $\det F$ must have to ensure the existence of a unique radial solution (i.e. the cube must continue to remain a cube) for any magnitude of radial stress acting on the cube. The function h proposed by Ciarlet and Geymonat satisfies these conditions. However, discontinuous equilibrium trajectories may

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occur, characterized by abruptly appearing non-symmetric deformations with increasing load, and a cube can instantaneously become a parallelepiped. Up to the load value for which the bifurcation in the radial solution is realized local monotonicity holds true. However, after exceeding this value, monotonicity no longer occurs on homogeneous deformations which, in turn, preserve the cube shape.

AMS subject classifications: 73G05, 73G10, 73H05

Key words: Neo-Hooke-Ciarlet-Geymonat, bifurcation, Rivlin's cube problem, invertibility, Hilbert-monotonicity, Biot stress-stretch relation.

1 Introduction

The theory of nonlinear elasticity is undoubtedly applicable in numerous contexts. However, depending on the specific phenomena we aim to analyze, different types of elastic energies come into play. Various materials exhibit different behaviors in terms of elasticity. In hyperelasticity, as considered here, stress is determined by the elastic energy density, making the selection of an energy function a crucial constitutive decision. The assumptions regarding the stress-strain relationship are referred to as constitutive requirements.

Therefore, one main task in hyperelasticity is to find an energy (or at least a family of energies) describing the behaviour of all, or at least a large class of materials. This question was raised by Clifford A. Truesdell (1919-2000) in "Das ungelöste Hauptproblem der endlichen Elastizitätstheorie, Zeit. Angew. Math. Mech. 36(3-4) (1956), 97–103". At present, however, there is no mathematical model in classical nonlinear elasticity which is capable of describing the correct physical or mechanical behaviour for every elastic material, especially for large strains and for which the existence of the minimizer of the corresponding variational problem or the Euler-Lagrange equations is ensured.

For different type of materials or for various behaviours which we wish to capture in the modelling process, we must choose an appropriate energy. In this contribution, we reconsider the classical compressible polyconvex Neo-Hooke-type energies (Hadamard materials) [31]

$$W_{\rm NH}(F)=\frac{\mu}{2}\|F\|^2+h(\det F), \tag{1.1}$$

where h is a convex function[1]. Here, we pay special attention to the Ciarlet-

[1] In order to have a stress free configuration, the function h must satisfy $3\mu/2+h'(1)=0$.