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The Biot Stress-Right Stretch Relation for the Compressible Neo-Hooke-Ciarlet-Geymonat Model and Rivlin's Cube Problem

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Abstract. The aim of the paper is to recall the importance of the study of invertibility and monotonicity of stress-strain relations for investigating the non-uniqueness and bifurcation of homogeneous solutions of the equilibrium problem of a hyperelastic cube subjected to equiaxial tensile forces. In other words, we reconsider a remarkable possibility in this nonlinear scenario: Does symmetric loading lead only to symmetric deformations or also to asymmetric deformations? If so, what can we say about monotonicity for these homogeneous solutions, a property which is less restrictive than the energetic stability criteria of homogeneous solutions for Rivlin's cube problem. For the Neo-Hooke type materials we establish what properties the volumetric function h depending on $\det F$ must have to ensure the existence of a unique radial solution (i.e. the cube must continue to remain a cube) for any magnitude of radial stress acting on the cube. The function h proposed by Ciarlet and Geymonat satisfies these conditions. However, discontinuous equilibrium trajectories may

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occur, characterized by abruptly appearing non-symmetric deformations with increasing load, and a cube can instantaneously become a parallelepiped. Up to the load value for which the bifurcation in the radial solution is realized local monotonicity holds true. However, after exceeding this value, monotonicity no longer occurs on homogeneous deformations which, in turn, preserve the cube shape.

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1 Introduction

The theory of nonlinear elasticity is undoubtedly applicable in numerous contexts. However, depending on the specific phenomena we aim to analyze, different types of elastic energies come into play. Various materials exhibit different behaviors in terms of elasticity. In hyperelasticity, as considered here, stress is determined by the elastic energy density, making the selection of an energy function a crucial constitutive decision. The assumptions regarding the stress-strain relationship are referred to as constitutive requirements.

Therefore, one main task in hyperelasticity is to find an energy (or at least a family of energies) describing the behaviour of all, or at least a large class of materials. This question was raised by Clifford A. Truesdell (1919-2000) in "Das ungelöste Hauptproblem der endlichen Elastizitätstheorie, Zeit. Angew. Math. Mech. 36(3-4) (1956), 97–103". At present, however, there is no mathematical model in classical nonlinear elasticity which is capable of describing the correct physical or mechanical behaviour for every elastic material, especially for large strains and for which the existence of the minimizer of the corresponding variational problem or the Euler-Lagrange equations is ensured.

For different type of materials or for various behaviours which we wish to capture in the modelling process, we must choose an appropriate energy. In this contribution, we reconsider the classical compressible polyconvex Neo-Hooke-type energies (Hadamard materials) [31]

$$W_{\rm NH}(F)=\frac{\mu}{2}\|F\|^2+h(\det F), \tag{1.1}$$

where h is a convex function[1]. Here, we pay special attention to the Ciarlet-

[1] In order to have a stress free configuration, the function h must satisfy $3\mu/2+h'(1)=0$.

Geymonat-type energy[2] [9, 10] for compressible materials, i.e. when the function h is of the form

$$h_{\rm CG}(x) = -\mu \log x + \frac{\lambda}{4}(x^2 - 2\log x - 1), \quad \lambda > 0 \tag{1.3}$$

with given positive constitutive parameters μ and λ. In the following, we shall refer to this model as the Neo-Hooke-Ciarlet-Geymonat model.

This paper revisits the issue of monotonicity and invertibility of a fundamental relationship: the Biot stress-right stretch tensor relation. We use recent analytical findings in this area as a first step towards addressing the non-symmetric bifurcation in the Rivlin cube problem associated with the Neo-Hooke-Ciarlet-Geymonat [9, 10] energy model. In order to understand these phenomena, we incorporate new insights pertaining to these constitutive equations. Studying the deformations of a uniformly loaded cube [28] is an important challenge within finite elasticity, revealing intriguing behaviours even for simple energy functions. Despite the straightforward mathematical setup of this equilibrium problem, its resolution can pose difficulties due to its inherent nonlinearity. Understanding the stability of solutions and the local monotonicity will add another layer of complexity.

The equilibrium problem of a cube under equitriaxial tensions was initially explored by Rivlin [28] and then by Rivlin and Beatty [29], by Reese and Wriggers [27], by Mihai, Woolley and Goriely [22] and by Tarantino [31], who all exposed multiple solutions, particularly in the realm of incompressible Neo-Hookean materials and compressible Neo-Hookean materials, respectively. Surprisingly, these solutions may lack symmetry, deviating from the (perhaps) expected behavior even under symmetric external loads. Rivlin further found that only one solution maintains full symmetry with the loading conditions, but becomes unstable under higher tensile loads. Ball and Schaeffer [3] have further explored into the case of incompressible Mooney-Rivlin materials, discovering the possibility of secondary bifurcations, a phenomenon absent in the Neo-Hookean scenario. They applied techniques from singularity theory to study the local behaviour around bifurcation points.

[2] Note that the original Ciarlet-Geymonat energy reads

$$W_{\rm CG}(F) = a\|F\|^2 + b\|{\rm Cof}F\|^2 + c(\det F)^2 - d\log(\det F), \tag{1.2}$$

where a,b,c,d are positive constants. This energy is polyconvex and agrees with the Saint-Venant energy quadratic in the Green-St Venant strain tensor $E = (C - \mathbb{1})/2$. For the original Ciarlet-Geymonat model it follows that the associated minimization problem has at least one solution by Ball's theorem [1]. Due to weak coercivity, the same is not known for (1.3).

The paper by Tarantino [31] aims to analyse both symmetric and asymmetric equilibrium configurations of bodies composed of general compressible isotropic materials. Special focus is put on exploring non-unique equilibrium states and relevant bifurcation phenomena. Building upon previous contributions by Ball [2] and Chen [6,7], in [31] a stability analysis is proposed to evaluate the stability of various homogeneous equilibrium branches, see also [30]. In the present paper, we rediscover some results obtained by Tarantino [31], but we rely on some pertinent analytical explanations, and present them in relation to our new results concerning the invertibility and monotonicity in nonlinear elasticity.

After a presentation of the problem and the general framework, in this article we establish results on the invertibility and monotonicity of stress-strain relations. These findings will subsequently be used for the Biot stress tensor-left stretch tensor relation and in the study of the Rivlin cube problem. For Neo-Hooke type materials we establish what properties the volumetric function h depending on $\det F$ in (1.1) must have to ensure the existence of a unique radial solution (i.e. the cube must continue to remain a cube) for any magnitude of radial stress acting on the cube.

In particular, we prove that the function $h \equiv h_{CG}$ in (1.1) defining the Ciarlet-Geymonat energies has these properties. For the Neo-Hooke-Ciarlet-Geymonat model, after identifying the radial solutions, we identify the existence of non-radial solutions (i.e. the cube turns into a parallelepiped) for the extension case. These solutions do not exist for the case of compression or if the magnitude of the forces does not exceed a certain critical value $\alpha^\flat$. Moreover, for radial and non-radial solutions the problem of monotonicity is studied using some new results similar to those studied in [26, last page in the Appendix] and [14–16]. Specifically, we prove that radial solutions ensure local monotonicity up to the critical value $\alpha^* \geq \alpha^\flat$ of the forces acting on the faces of the cube. This is where bifurcation occurs, i.e. the solution is no longer locally unique and beyond this value monotonicity no longer occurs in radial solutions. In this regard, we prove that the critical value α^* corresponds to the critical values of the stretch for which the invertibility in terms of the principal stress-principal stretch relation is lost in radial solutions.

Starting from a value of the magnitude $\alpha^\flat$ of the forces (less than the critical value that produces the bifurcation) there are two other types of non-radial solutions (other types are obtained by permutations of them). We show that these types of non-radial solutions cannot all have different eigenvalues but certainly at least two are equal. Returning to the type of non-radial solutions, both appear in a discontinuous manner for a value $\alpha^\flat$ of the magnitude of the forces and then depend continuously on the intensity of the forces. One class of non-radial solutions continuously moves towards and through the bifurcation branch, while the other moves away from the bifurcation point. Numerical tests have shown that, while the first class does not ensure local monotonicity, the second one does enjoy monotonicity. It is for this reason that the latter solution meets the physical expectations in a better way.

2 Statement of the problem

Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. A mapping $\varphi:\Omega\to\mathbb{R}$ describes the deformation of the domain Ω. The domain Ω is called the initial (undeformed) configuration, while its image $\Omega_c:=\varphi(\Omega)$ is called the actual (deformed) configuration. Each of these configurations could be considered as a reference configuration, depending on the practical problem we solve or model. Since we do not allow for self-intersection of the material, there exists the inverse mapping $\varphi^{-1}:\Omega_c\to\Omega$ from the deformed configuration to its initial configuration. Therefore, imposing the preservation of orientation, the deformation gradient defined by $F:=\mathrm{D}\varphi\in\mathbb{R}^{3\times 3}$ satisfies $F\in\mathrm{GL}^+(3)$, i.e. $J=\det F>0$. For further notation, the reader is referred to Appendix A.

From a geometric or analytic point of view, this would suffice for a complete description of the deformation. However, in elasticity theory we assume that the domain Ω is filled by an elastic body. Thus, the aim is to take into account the physical response of the body, meaning the constitutive relation between stress (internal forces) and strain (amount of deformation). In the context of nonlinear hyperelasticity, where generalized convexity properties have an especially long and rich history [1, 18, 19], the material behaviour of an elastic solid is described by a potential energy function $W:\mathrm{GL}^+(3)\to\mathbb{R}$, $F\mapsto W(F)$ defined on the group $\mathrm{GL}^+(3)$ of invertible matrices with positive determinants. In hyperelasticity, the tensor which describes the force of the deformed material per original area (stress) is the first Piola-Kirchhoff stress tensor, denoted here by $S_1(F)$, and the stress-strain relation is described by the energy density potential $W:\mathrm{GL}^+(3)\to\mathbb{R}$, through $S_1=\mathrm{D}_F[W(F)]$. We assume that the material is homogeneous. The elastic energy potential $W:\mathrm{GL}^+(3)\to\mathbb{R}$ is also assumed to be objective (or frame-indifferent) as well as isotropic, i.e. it to satisfy $W(Q_1FQ_2)=W(F)$ for all $F\in\mathrm{GL}^+(3)$ and all $Q_1,Q_2\in\mathrm{SO}(3)$. Hence, $W(F)=\widehat{W}(U)$, where U is the right stretch tensor, i.e. the unique element of $\mathrm{Sym}^{++}(3)$ for which $U^2=C:=F^\top F$; here and throughout, $\mathrm{Sym}^{++}(3)$ denotes the positive definite, symmetric tensors.

In the absence of body forces, the general boundary value problem is to find the solution φ of the equilibrium equation

$$\operatorname{Div} S_1(\mathrm{D}\varphi) = 0 \quad \text{in } \Omega \subset \mathbb{R}^3, \tag{2.1}$$

where $S_1(\mathrm{D}\varphi)$ is the first Piola-Kirchhoff stress tensor, subject to the boundary conditions

$$S_1.N = \widehat{s}_1. \tag{2.2}$$

Here, N is the unit normal at the boundary $\partial\Omega$ and the vector $\widehat{s}_1$ is given.

In this article, we study the invertibility of the stress-stretch relation, the monotonicity and the bifurcation problem for a dead loading problem. In Rivlin's cube problem [20, p. 15] (see also [3,29,31,33]) the unit cube is subjected to equal and opposite normal dead loads on all faces[3]. Dead loading is a simple example of a traction boundary condition where $\widehat{s}_1(x)$ is a constant vector at each point of the boundary $\partial\Omega$. Thus, the boundary conditions on the face of the unit coordinate normal N_i are

$$S_1.N_i = \alpha N_i, \quad i = 1,2,3 \tag{2.3}$$

with α indicating the amount of load.

A minimizer $\varphi \in C^2(\Omega)$ of the total energy functional given by

$$\begin{aligned} I(\varphi) &= \int_\Omega W(\mathrm{D}\varphi)\, dV - \int_{\partial\Omega} \langle S_1.N, \varphi\rangle\, dA \\ &= \int_\Omega W(\mathrm{D}\varphi)\, dV - \int_{\partial\Omega} \langle \widehat{s}_1, \varphi\rangle\, dA \end{aligned} \tag{2.4}$$

is a solution of the boundary value problem given by (2.1) and (2.2).

For Rivlin's cube problem, the body is subjected to a uniform load on the boundary $S_1.N_i = \alpha N_i$, $i = 1,2,3$. An application of the divergence theorem yields that the total energy functional is given by

$$\begin{aligned} I(\varphi) &= \int_\Omega W(\mathrm{D}\varphi) dV - \int_{\partial\Omega} \alpha \langle N, \varphi\rangle dA \\ &= \int_\Omega [W(\mathrm{D}\varphi) - \alpha \operatorname{div}\varphi]\, dV = \int_\Omega [W(\mathrm{D}\varphi) - \alpha \operatorname{tr}(\mathrm{D}\varphi)]\, dV. \end{aligned} \tag{2.5}$$

A deformation φ is homogeneous if the deformation gradient $\mathrm{D}\varphi$ is constant in Ω. For a homogeneous deformation the equilibrium equations are immediately satisfied, while the boundary conditions give rise to a system of nonlinear algebraic equations.

We recall that [32, p. 144]

$$T_{\text{Biot}} = R^\top S_1, \tag{2.6}$$

where $T_{\text{Biot}} = R^\top S_1(F) = \mathrm{D}_U[\widehat{W}(U)]$ is the symmetric Biot stress tensor and R is

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[3]"Rivlin's solutions have been known for nearly half a century. Nevertheless, we have yet to find an experiment that demonstrates these solutions" [8].

the orthogonal matrix of the polar decomposition $F=RU=VR$, see [5,11,25]; here $V=\sqrt{FF^\top}$ is the left stretch tensor. It is known that

$T_{\rm Biot}$ is symmetric and represents "the principal forces acting in the reference system".

Since no rotations are present in the cube problem ($R=\mathbb{1}$), we are able to rephrase Rivlin's cube problem [26] as the algebraic nonlinear system

$$T_{\rm Biot}(U)N_i=\alpha N_i \iff \big(T_{\rm Biot}(U)-\alpha\cdot\mathbb{1}\big)N_i=0, \quad i=1,2,3, \tag{2.7}$$

where N_i are the linearly independent normals to the faces. Here, we adopt the convention that only homogeneous solutions U are considered. Therefore, (2.7) is equivalent to

$$T_{\rm Biot}(U)=\alpha\cdot\mathbb{1}. \tag{2.8}$$

For the Neo-Hooke model, Rivlin has shown that the problem (2.8) admits several homogeneous solutions[4], see Fig. 1, and for a certain load parameter α the (always existing) homogeneous radial solution $U=\beta^+\mathbb{1},\beta^+>0$, becomes un-

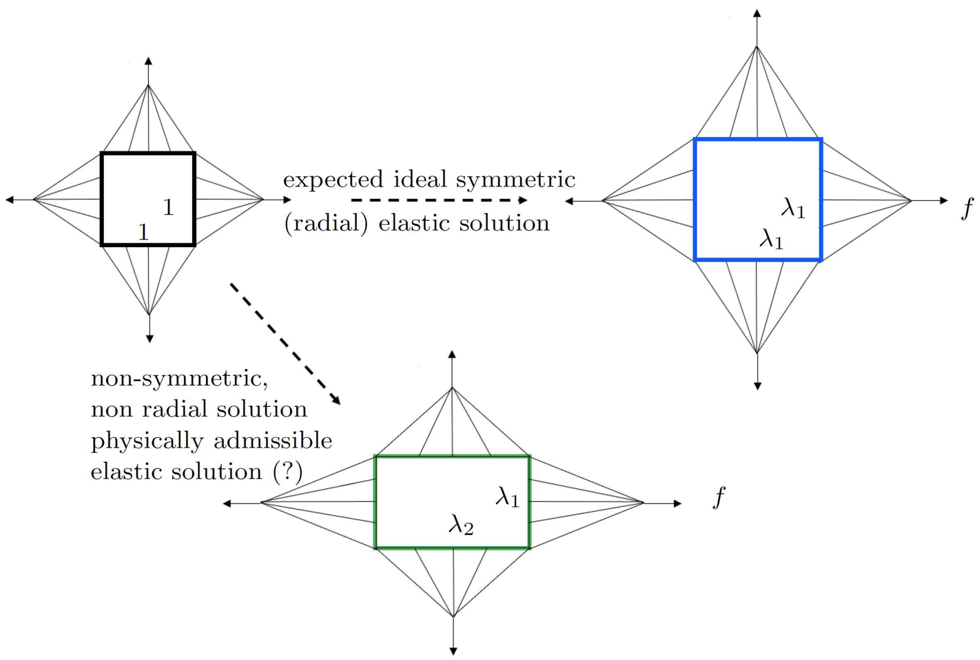

Figure 1: Bifurcation for the compressible Rivlin's cube dead load problem: equal and opposite normal dead loads on all faces.

[4]The radial solution is an abbreviation for the equitriaxial stretching $\lambda_1=\lambda_2=\lambda_3=\beta^+$.

stable. Thus, an initially homogeneous and perfectly isotropic material would not behave as we expect intuitively from an isotropic material. Whether this can be really observed in experiments remains an open question. Discussing the related notion of Kearsley's instability, Batra *et al.* [4, pp. 710-711] "must conclude, rather prosaically, that Treloar's observation of two different stretches for equal loads is nothing else but another example of a notorious quality of rubber, namely the difficulty of quantitative reproducibility of rubber data and the unreliability of exact numbers obtained from rubber experiments", but provide new experimental data that indeed supports Kearsley's claims of instabilities in rubber sheets (cf. [13]).

For Neo-Hookean materials, the problem (2.8) admits several non-symmetric homogeneous solutions $U\in\mathrm{Sym}^{++}(3)$. Based on (2.6) we are able to investigate several equivalent statements in terms of different stress tensors. Since in Rivlin's cube problem there is no cause for a non-symmetric response, it is questionable if there should be any non-symmetric response[5].

Ideally, we aim at the radial solution to be locally unique among all other solutions. In particular, we insist on the logical rule that there is "no effect without a corresponding cause". Since in Rivlin's cube problem there is no cause for a non-symmetric response, there should be no admissible non-symmetric response[6]. If this is or is not the case, depends on the chosen constitutive relation.

If there is a radial solution $U=\beta^+\mathbb{1}$ of Rivlin's cube problem then invertibility and monotonicity of $T_{\mathrm{Biot}}:\mathrm{Sym}^{++}(3)\to\mathrm{Sym}(3)$ suffice to exclude symmetric bifurcations altogether, as it will be seen in the rest of the paper.

3 Constitutive requirements in nonlinear elasticity

3.1 Invertibility

We consider the general isotropic constitutive equation

$$\Sigma=\Sigma(U),\quad \Sigma:\mathrm{Sym}^{++}(3)\;\to\;\mathrm{Sym}(3), \tag{3.1}$$

where Σ is some symmetric stress tensor and $U\in\mathrm{Sym}^{++}(3)$ is the stretch tensor. We are then interested in the following two important questions:

[5]Of course, rubber is not at all incompressible under high pressure; rather, for moderate pressure, rubber "tries" to respond in a way which preserves volume due to a comparatively low shear modulus compared to the bulk modulus

[6]"Experimentally" observed non-symmetric bifurcations seem to be inevitably accompanied by permanent deformations [4].

i) (Surjectivity) Given any symmetric tensor $\Sigma\in\mathrm{Sym}(3)$, does there exist a positive definite tensor $U\in\mathrm{Sym}^{++}(3)$ such that $\Sigma=\Sigma(U)$?

ii) (Injectivity) For a given symmetric tensor $\Sigma\in\mathrm{Sym}(3)$, does there exist at most one $U\in\mathrm{Sym}^{++}(3)$ such that $\Sigma=\Sigma(U)$?

It is clear that when an idealised model is proposed (hence, no elasto-plastic response is expected) the first requirement seems mandatory. The second requirement is the first step in order to exclude bifurcation [28] for a dead loading problem.

We have observed a possible way to study the invertibility of the map $U\mapsto T_{\mathrm{Biot}}(U)$. Since, in general, it is not easy to work with tensors (matrices) in three dimensions, we consider the singular values (the principal stretches) $\lambda_1,\lambda_2,\lambda_3$ of F, i.e. the positive eigenvalues of U. If $\Sigma_f:\mathrm{Sym}(3)\to\mathrm{Sym}(3)$ is an isotropic tensor function satisfying

$$\Sigma_f\big(Q^\top\cdot\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)\cdot Q\big)=Q^\top\cdot\Sigma_f\big(\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)\big)\cdot Q, \tag{3.2}$$

then

$$\begin{aligned}\Sigma_f(U):=&\,\Sigma_f\big(\underbrace{Q^\top\cdot\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)\cdot Q}_{S\in\mathrm{Sym}(3)}\big)\\ =&\,\underbrace{Q^\top\cdot\mathrm{diag}\big(f(\lambda_1,\lambda_2,\lambda_3)\big)\cdot Q}_{\Sigma_f(S)\in\mathrm{Sym}(3)},\quad \forall Q\in\mathrm{O}(3)\end{aligned} \tag{3.3}$$

with a vector-function $f=(f_1,f_2,f_3):\mathbb{R}^3\to\mathbb{R}^3$ which fulfills

$$f_i(\lambda_{\pi(1)},\lambda_{\pi(2)},\lambda_{\pi(3)})=f_{\pi(i)}(\lambda_1,\lambda_2,\lambda_3)$$

for any permutation $\pi:\{1,2,3\}\to\{1,2,3\}$. Here, $\mathrm{Sym}(3)$ denotes the space of symmetric 3×3 matrices, $\mathrm{O}(3)$ is the orthogonal group and $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$ is the diagonal matrix with diagonal entries $\lambda_1,\lambda_2,\lambda_3$. The permutation symmetry implies and it is implied by isotropy.

Indeed, in many situations the stress-strain relations are characterized by the relation between their corresponding principal values, i.e. by the relations between the principal stretches $\lambda_1,\lambda_2,\lambda_3$ and the principal forces (principal Biot-stresses)

$$T_i=\frac{\partial g(\lambda_1,\lambda_2,\lambda_3)}{\partial\lambda_i},\quad i=1,2,3. \tag{3.4}$$

In (3.4), $g\colon \mathbb{R}^3_+ \to \mathbb{R}$ is the unique permutation symmetric function of the singular values of U (principal stretches) such that $W(F) = \widehat{W}(U) = g(\lambda_1,\lambda_2,\lambda_3)$, where $\mathbb{R}^3_+ = (0,\infty)\times(0,\infty)\times(0,\infty)$, and

$$\widehat{T} := (T_1, T_2, T_3)^\top. \tag{3.5}$$

The functions f and Σ_f related by Eq. (3.3) share a number of properties related to invertibility and monotonicity.

Theorem 3.1. *Let $f:\mathbb{R}^3_+ \to \mathbb{R}^3$ be symmetric.*

i) The function $\Sigma_f:\mathrm{Sym}^{++}(3) \to \mathrm{Sym}(3)$ is injective if and only if f is injective.

ii) The function $\Sigma_f:\mathrm{Sym}^{++}(3) \to \mathrm{Sym}(3)$ is surjective if and only if f is surjective.

In particular, Σ_f is invertible if and only if f is invertible.

In particular, using the result by Katriel [17] (see also [12]) proving the global homeomorphism theorem of Hadamard, we obtain a sufficient criterion for the global invertibility of an isotropic tensor function.

Proposition 3.1. *Assume that $\widetilde{\Sigma}_{\widetilde{f}}$:$\mathrm{Sym}(3) \to \mathrm{Sym}(3)$ is an isotropic C^1-function defined by the vector-function $\widetilde{f}:\mathbb{R}^3 \to \mathbb{R}^3$ such that*

1) $\mathrm{D}\widetilde{f}(x_1,x_2,x_3)$ is invertible for any $(x_1,x_2,x_3)\in\mathbb{R}^3$,

2) $\|\widetilde{f}(x_1,x_2,x_3)\|_{\mathbb{R}^3} \to \infty$ as $\|(x_1,x_2,x_3)\|_{\mathbb{R}^3} \to \infty$.

Then $U \mapsto \widetilde{\Sigma}_{\widetilde{f}}(U)$ is a global diffeomorphism from $\mathrm{Sym}(3)$ *to* $\mathrm{Sym}(3)$.

Let us remark that Proposition 3.1 is not directly applicable to Σ_f:$\mathrm{Sym}^{++}(3) \to \mathrm{Sym}(3)$. However, we have the following corollary to Katriel's result:

Corollary 3.1. *Assume that $\Sigma_f:\mathrm{Sym}^{++}(3) \to \mathrm{Sym}(3)$ is an isotropic C^1-function such that*

1) $\mathrm{D}(f\circ\exp)(x_1,x_2,x_3)$ is invertible[7] for any $(x_1,x_2,x_3)\in\mathbb{R}^3$,

2) $\|(f\circ\exp)(x_1,x_2,x_3)\|_{\mathbb{R}^3} \to \infty$ as $\|(x_1,x_2,x_3)\|_{\mathbb{R}^3} \to \infty$.

Then $U \mapsto \Sigma_f(U)$ is a global diffeomorphism from $\mathrm{Sym}^{++}(3)$ *to* $\mathrm{Sym}(3)$.

[7]Here, $\exp(x_1,x_2,x_3) = (\exp x_1, \exp x_2, \exp x_3)$ for any $(x_1,x_2,x_3)\in\mathbb{R}^3$.

Proof. Let us consider $\widetilde{\Sigma}_{\widetilde{f}}:\mathrm{Sym}(3)\to\mathrm{Sym}(3)$ defined by

$$\begin{aligned} &\widetilde{\Sigma}_{\widetilde{f}}(S):=(\Sigma_f\circ\exp)(S), \quad \forall S\in\mathrm{Sym}(3)\\ \Longleftrightarrow\ &\Sigma_{\widetilde{f}}(\log U):=\Sigma_f(U), \qquad \forall U\in\mathrm{Sym}^{++}(3), \end{aligned} \tag{3.6}$$

where

$$\log U=\sum_{i=1}^{3}\log\lambda_i\, N_i\otimes N_i$$

with N_i the eigenvectors of U and λ_i the eigenvalues of U, is the Hencky strain tensor [23,24]. The function $\widetilde{f}$ defining $\widetilde{\Sigma}_{\widetilde{f}}$ is $\widetilde{f}:=f\circ\exp:\mathbb{R}^3\to\mathbb{R}^3$. Now, Katriel's result applied to $\widetilde{\Sigma}_{\widetilde{f}}$ shows that if $\mathrm{D}\widetilde{f}(x_1,x_2,x_3)$ is invertible for any $(x_1,x_2,x_3)\in\mathbb{R}^3$ and

$$\|\widetilde{f}(x_1,x_2,x_3)\|_{\mathbb{R}^3}\ \to\ \infty \quad \text{as} \quad \|(x_1,x_2,x_3)\|_{\mathbb{R}^3}\ \to\ \infty,$$

then $U\mapsto\widetilde{\Sigma}_{\widetilde{f}}(U)$ is a global diffeomorphism from $\mathrm{Sym}(3)$ to $\mathrm{Sym}(3)$. Then, since the matrix logarithm function $\log:\mathrm{Sym}^{++}(3)\to\mathrm{Sym}(3)$ is a global diffeomorphism, $U\mapsto\Sigma_f(U)=(\widetilde{\Sigma}_{\widetilde{f}}\circ\log)(U)$ must be a global diffeomorphism as well. ☐

Corollary 3.2. *Assume that $\Sigma_f:\mathrm{Sym}^{++}(3)\to\mathrm{Sym}(3)$ is an isotropic C^1-function such that*

(1) $\mathrm{D}f(\lambda_1,\lambda_2,\lambda_3)$ is invertible for any $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3_+$,

(2) $\|f(\lambda_1,\lambda_2,\lambda_3)\|_{\mathbb{R}^3}\to\infty$ as $\|(\log\lambda_1,\log\lambda_2,\log\lambda_3)\|_{\mathbb{R}^3}\to\infty$.

Then $U\mapsto\Sigma_f(U)$ is a global diffeomorphism from $\mathrm{Sym}^{++}(3)$ to $\mathrm{Sym}(3)$.

Proof. First, by using the chain rule and the invertibility of $D\exp$, we observe that the assumption that $\mathrm{D}f(\lambda_1,\lambda_2,\lambda_3)$ is invertible for any $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3_+$ is equivalent to the invertibility of $\mathrm{D}(f\circ\exp)(x_1,x_2,x_3)$ for any $(x_1,x_2,x_3)\in\mathbb{R}^3$.

Consider now the condition $\|(x_1,x_2,x_3)\|_{\mathbb{R}^3}\to\infty$. We will prove that under assumption (2) in the corollary, it follows that $\|(f\circ\exp)(x_1,x_2,x_3)\|_{\mathbb{R}^3}\to\infty$. Indeed, let $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3_+$ be such that $x_i=\log\lambda_i$ for $i=1,2,3$. Then,

$$\|(\log\lambda_1,\log\lambda_2,\log\lambda_3)\|_{\mathbb{R}^3}\ \to\ \infty.$$

From (2), this implies that

$$\|f(\lambda_1,\lambda_2,\lambda_3)\|=\|(f\circ\exp)(x_1,x_2,x_3)\|\ \to\ \infty.$$

Therefore, the requirements of Corollary 3.1 are satisfied, and this implies that $U\mapsto\Sigma_f(U)$ is a global diffeomorphism from $\mathrm{Sym}^{++}(3)$ to $\mathrm{Sym}(3)$. ☐

3.2 Hilbert-monotonicity

For our purposes, we now recall some related notions of monotonicity.

Definition 3.1 ([21]). *A tensor function* $\Sigma_f : \mathrm{Sym}^{++}(3) \to \mathrm{Sym}(3)$ *is called strictly Hilbert-monotone if*

$$\langle \Sigma_f(U) - \Sigma_f(\overline{U}), U - \overline{U} \rangle_{\mathbb{R}^{3\times 3}} > 0, \quad \forall U \neq \overline{U} \in \mathrm{Sym}^{++}(3). \tag{3.7}$$

We refer to this inequality as strict Hilbert-space matrix-monotonicity of the tensor function Σ_f*. A tensor function* $\Sigma_f : \mathrm{Sym}^{++}(3) \to \mathrm{Sym}(3)$ *is called Hilbert-monotone if*

$$\langle \Sigma_f(U) - \Sigma_f(\overline{U}), U - \overline{U} \rangle_{\mathbb{R}^{3\times 3}} \geq 0, \quad \forall U, \overline{U} \in \mathrm{Sym}^{++}(3). \tag{3.8}$$

Definition 3.2 ([21]). *A vector function* $f : \mathbb{R}^3_+ \to \mathbb{R}^3$ *is strictly vector monotone if*

$$\langle f(\lambda) - f(\overline{\lambda}), \lambda - \overline{\lambda} \rangle_{\mathbb{R}^3} > 0, \quad \forall \lambda \neq \overline{\lambda} \in \mathbb{R}^3_+, \tag{3.9}$$

and it is vector monotone if

$$\langle f(\lambda) - f(\overline{\lambda}), \lambda - \overline{\lambda} \rangle_{\mathbb{R}^3} \geq 0, \quad \forall \lambda, \lambda \in \mathbb{R}^3_+. \tag{3.10}$$

Definition 3.3. *A continuously differentiable tensor function* $\Sigma_f : \mathrm{Sym}^{++}(3) \to \mathrm{Sym}(3)$ *is called strongly Hilbert-monotone if*

$$\langle \mathrm{D}\Sigma_f . H(U), H \rangle > 0, \quad \forall U \in \mathrm{Sym}^{++}(3), \quad H \in \mathrm{Sym}(3).$$

Definition 3.4 ([21]). *A continuously differentiable vector function* $f : \mathbb{R}^3_+ \to \mathbb{R}^3$ *is called strongly vector monotone if*

$$\langle \mathrm{D}f(\lambda).h, h \rangle > 0, \quad \forall \lambda \in \mathbb{R}^3_+, \quad h \in \mathbb{R}^3.$$

Note that $\mathrm{D}f(\lambda_1,\lambda_2,\lambda_3)$ in itself might not be symmetric. However, for $T_i = \partial g(\lambda_1,\lambda_2,\lambda_3)/\partial \lambda_i, i = 1,2,3$,

$$\mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3) = \mathrm{D}^2 g(\lambda_1,\lambda_2,\lambda_3) := \left(\frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} \right)_{i,j=1,2,3} \in \mathrm{Sym}(3). \tag{3.11}$$

In a forthcoming paper, we discuss the following result, thereby expanding on Ogden's work [26, last page in the Appendix], following Hill's seminal contributions [14–16]:

Theorem 3.2. *A sufficiently regular symmetric function* $f:\mathbb{R}^3_+\to\mathbb{R}^3$ *is (strictly/strongly) vector-monotone if and only if* Σ_f *is (strictly/strongly) matrix-monotone.*

Hence, the following holds true for hyperelasticity, assuming sufficient regularity:

$$
\begin{aligned}
& U\mapsto T_{\rm Biot}(U) \quad \text{Hilbert-monotone}\\
\iff\ & (\lambda_1,\lambda_2,\lambda_3)\ \mapsto\ \widehat{T}(\lambda_1,\lambda_2,\lambda_3) \quad \text{vector monotone}\\
\iff\ & {\rm D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)\in{\rm Sym}^+(3), \quad \forall(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3_+,
\end{aligned}
$$

$$
\begin{aligned}
& U\mapsto T_{\rm Biot}(U) \quad \text{strictly Hilbert-monotone}\\
\iff\ & (\lambda_1,\lambda_2,\lambda_3)\ \mapsto\ \widehat{T}(\lambda_1,\lambda_2,\lambda_3) \quad \text{strictly vector monotone,}
\end{aligned}
$$

$$
\begin{aligned}
& U\mapsto T_{\rm Biot}(U) \quad \text{strongly Hilbert-monotone}\\
\iff\ & (\lambda_1,\lambda_2,\lambda_3)\ \mapsto\ \widehat{T}(\lambda_1,\lambda_2,\lambda_3) \quad \text{strongly vector monotone}\\
\iff\ & {\rm D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)\in{\rm Sym}^{++}(3), \quad \forall(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3_+.
\end{aligned}
$$

Note that the monotonicity conditions and the invertibility condition are global conditions. Conversely, the conditions $\det{\rm D}f(\lambda_1,\lambda_2,\lambda_3)\neq 0$ – which is equivalent to f being a local diffeomorphism – as well as ${\rm D}\widehat{T}\in{\rm Sym}^{++}(3)$ are only local conditions.

3.3 Energetic stability

In the following, we employ the stability criterion

$$\langle {\rm D}^2_F W(F).H,H\rangle\geq 0, \quad \forall H\in\mathbb{R}^{3\times 3} \tag{3.12}$$

for the hyperelastic energy potential W, which ensures material stability under so-called soft loads [7]. In terms of the singular values, the condition (3.12) holds at $F\in{\rm GL}^+(3)$ if and only if [6]

$$\left.\frac{\partial g/\partial\lambda_i-\epsilon_{ij}\partial g/\partial\lambda_j}{\lambda_i-\epsilon_{ij}\lambda_j}\right|_{\lambda_i=\lambda_i^*}\geq 0 \quad \text{holds} \quad \forall i,j=1,2,3, \quad i\neq j \quad \text{(no sum)}, \tag{3.13}$$

and the Hessian matrix of g, i.e. ${\rm D}^2g=(\partial^2 g/(\partial\lambda_i\partial\lambda_j))|_{\lambda_i=\lambda_i^*}$ is positive semi-definite, where

$$\epsilon_{ij}=\begin{cases} 1, & \text{if} \quad \{i,j\}=\{1,2\} \quad \text{or} \quad \{2,3\} \quad \text{or} \quad \{3,1\},\\ -1, & \text{otherwise,}\end{cases} \tag{3.14}$$

and λ_i^* are the singular values of F. If two singular values λ_i^* and $\lambda_j^*, i\neq j$, are equal, the inequalities in (3.13) are interpreted in terms of their limits $\lambda_i^*\to\lambda_j^*$; for instance, in the points $(\lambda_1^*,\lambda_1^*,\lambda_3^*)=(\lambda^*,\lambda^*,\lambda_3^*)$ with $\lambda_3^*\neq\lambda^*$, the energetic stability criterion (3.12) is satisfied if and only if

$$\begin{aligned}
&\left(\frac{\partial^2 g}{\partial\lambda_1^2}-\frac{\partial^2 g}{\partial\lambda_1\partial\lambda_2}\right)\Big|_{\lambda_1=\lambda_2=\lambda^*,\lambda_3=\lambda_3^*}\geq 0,\\
&\left(\frac{\partial g/\partial\lambda_2-\partial g/\partial\lambda_3}{\lambda_2-\lambda_3}\right)\Big|_{\lambda_1=\lambda_2=\lambda^*,\lambda_3=\lambda_3^*}\geq 0,\\
&\left(\frac{\partial g/\partial\lambda_2+\partial g/\partial\lambda_1}{2\lambda_1}\right)\Big|_{\lambda_1=\lambda_2=\lambda^*,\lambda_3=\lambda_3^*}\geq 0,\\
&\left(\frac{\partial g/\partial\lambda_2+\partial g/\partial\lambda_3}{\lambda_2+\lambda_3}\right)\Big|_{\lambda_1=\lambda_2=\lambda^*,\lambda_3=\lambda_3^*}\geq 0,
\end{aligned}\tag{3.15}$$

and the Hessian matrix $\mathrm{D}^2 g=(\partial^2 g/(\partial\lambda_i\partial\lambda_j))|_{\lambda_1=\lambda_2=\lambda^*,\lambda_3=\lambda_3^*}$ is positive semi-definite.

We also remark that the positive semi-definiteness of the Hessian matrix of g is equivalent to the positive semi-definiteness of $\mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)=\mathrm{D}^2 g(\lambda_1,\lambda_2,\lambda_3)$. Since the stability implies the positive semi-definiteness of $\mathrm{D}^2 g(\lambda_1,\lambda_2,\lambda_3)=\mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)$, the stability implies the monotonicity of $\mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)$.

4 Invertibility and monotonicity of the Biot stress-stretch relation for the compressible Neo-Hooke-Ciarlet-Geymonat energy

In the following, we will reduce the Neo-Hooke-Ciarlet-Geymonat energy to its one-parameter version

$$\begin{aligned}
W_{\mathrm{CG}}^M(F)&=\frac{1}{\mu}W_{\mathrm{CG}}(F)\\
&=\frac{1}{2}\|F\|^2+\left[-\log\det F+\left(\frac{M}{4}-\frac{1}{6}\right)\left((\det F)^2-2\log\det F-1\right)\right]
\end{aligned}\tag{4.1}$$

with $M:=(\lambda+2\mu/3)/\mu>2/3$. All the stresses considered in the following will be related to this one parameter energy. In terms of the singular values, W_{CG}^M admits the representation

$$\begin{aligned} W_{\rm CG}^M(F) &= g(\lambda_1,\lambda_2,\lambda_3) \\ &= \frac{1}{2}\bigg[\frac{1}{6}(3M-2)\big(\lambda_1^2\lambda_2^2\lambda_3^2 - 2\log(\lambda_1\lambda_2\lambda_3)-1\big) \\ &\qquad -2\log(\lambda_1\lambda_2\lambda_3)+\lambda_1^2+\lambda_2^2+\lambda_3^2\bigg]. \end{aligned} \tag{4.2}$$

The corresponding first Piola-Kirchhoff stress tensor is given by

$$\begin{aligned} S_1 &= F + \frac{1}{\mu}\,(h_{\rm CG}^M)'(\det F)\cdot {\rm Cof} F \\ &= F + \left[\left(\frac{M}{2}-\frac{1}{3}\right)\left(\det F - \frac{1}{\det F}\right) - \frac{1}{\det F}\right]\cdot {\rm Cof} F, \end{aligned} \tag{4.3}$$

where

$$h_{\rm CG}^M := \frac{1}{\mu} h_{\rm CG} = -\log x + \frac{\lambda}{4\mu}(x^2 - 2\log x - 1). \tag{4.4}$$

The Biot stress tensor defined by $W_{\rm CG}^M(F)$ is

$$\begin{aligned} T_{\rm Biot}(U) &= {\rm D}_U W_{\rm CG}^M(U) = R^\top S_1 \\ &= U + \frac{1}{\mu}\,(h_{\rm CG}^M)'(\det F)(\det U)\cdot \det U \cdot U^{-1} \\ &= U + \left[\left(\frac{M}{2}-\frac{1}{3}\right)\left(\det U - \frac{1}{\det U}\right) - \frac{1}{\det U}\right]\cdot \det U\cdot U^{-1}, \end{aligned} \tag{4.5}$$

while the principal Biot stresses are given by

$$\begin{aligned} T_1 &= \lambda_1 - \frac{1}{\lambda_1} + \left(\frac{M}{2}-\frac{1}{3}\right)\left(\lambda_1\lambda_2^2\lambda_3^2 - \frac{1}{\lambda_1}\right), \\ T_2 &= \lambda_2 - \frac{1}{\lambda_2} + \left(\frac{M}{2}-\frac{1}{3}\right)\left(\lambda_1^2\lambda_2\lambda_3^2 - \frac{1}{\lambda_2}\right), \\ T_3 &= \lambda_3 - \frac{1}{\lambda_3} + \left(\frac{M}{2}-\frac{1}{3}\right)\left(\lambda_1^2\lambda_2^2\lambda_3 - \frac{1}{\lambda_3}\right). \end{aligned} \tag{4.6}$$

We compute

$${\rm D}\widehat{T} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{pmatrix}, \tag{4.7}$$

where

$$D_{11}=\frac{1}{\lambda_1^2}+1+\left(\frac{M}{2}-\frac{1}{3}\right)\left(\frac{1}{\lambda_1^2}+\lambda_2^2\lambda_3^2\right),\quad D_{12}=2\left(\frac{M}{2}-\frac{1}{3}\right)\lambda_1\lambda_2\lambda_3^2,$$
$$D_{13}=2\left(\frac{M}{2}-\frac{1}{3}\right)\lambda_1\lambda_2^2\lambda_3,\qquad D_{22}=\frac{1}{\lambda_2^2}+1+\left(\frac{M}{2}-\frac{1}{3}\right)\left(\lambda_1^2\lambda_3^2+\frac{1}{\lambda_2^2}\right),$$
$$D_{23}=2\left(\frac{M}{2}-\frac{1}{3}\right)\lambda_1^2\lambda_2\lambda_3,\qquad D_{33}=\frac{1}{\lambda_3^2}+1+\left(\frac{M}{2}-\frac{1}{3}\right)\left(\lambda_1^2\lambda_2^2+\frac{1}{\lambda_3^2}\right),$$

and remark that

$$\det \mathrm{D}\widehat{T}(1,1,1)=12M, \tag{4.8}$$

which is strictly positive for all $M>0$. However, we find that for all $M>2/3$ there exists $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}_+^3$ such that

$$\det \mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)=0. \tag{4.9}$$

A quick numerical check reveals that the Biot stress-stretch relation is in general not invertible, see Fig. 2. However, we may equally show this analytically. Indeed, for each material of the form (4.1) with $M>2/3$ we have

$$\begin{aligned}&\det \mathrm{D}\widehat{T}(\lambda_1,\lambda_1,\lambda_1)\\ &=\frac{\left[(2-3M)\lambda_1^6+6\lambda_1^2+4+3M\right]^2\left[5(3M-2)\lambda_1^6+6\lambda_1^2+4+3M\right]}{216\lambda_1^6},\end{aligned} \tag{4.10}$$

and therefore, $U\mapsto T_{\mathrm{Biot}}(U)$ loses differentiable invertibility in $U=\lambda^*\mathbb{1}$, where λ_1 is a solution of the equation (see Fig. 3)

$$(-3M+2)\lambda_1^6+6\lambda_1^2+4+3M=0. \tag{4.11}$$

In Fig. 3, for fixed M, the solution is the intersection of the red line to the blue curve. However, the analytical proof of the existence and uniqueness of the solution λ^* of (4.11) is also possible.

Proposition 4.1. *For any $M>2/3$ the Biot stress-stretch relation $U\mapsto T_{\mathrm{Biot}}(U)$ for the Neo-Hooke-Ciarlet-Geymonat energy is in general not a diffeomorphism.*

Proof. Let us consider the function

$$s:(0,\infty)\to\mathbb{R},\quad s(x)=(-3M+2)x^3+6x+3M+4. \tag{4.12}$$

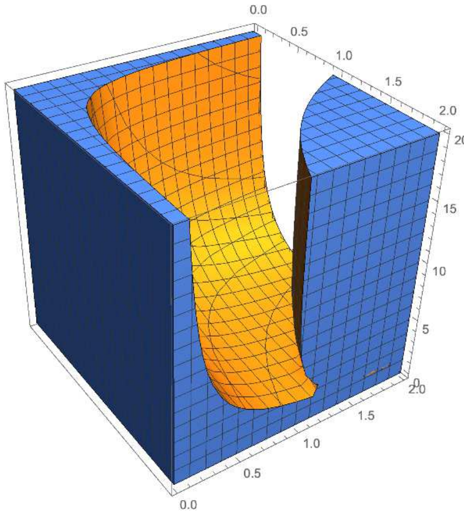

Figure 2: For $M=1$, the region of those $(\lambda_1,\lambda_2,\lambda_3)$ for which $\det \mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)\neq 0$.

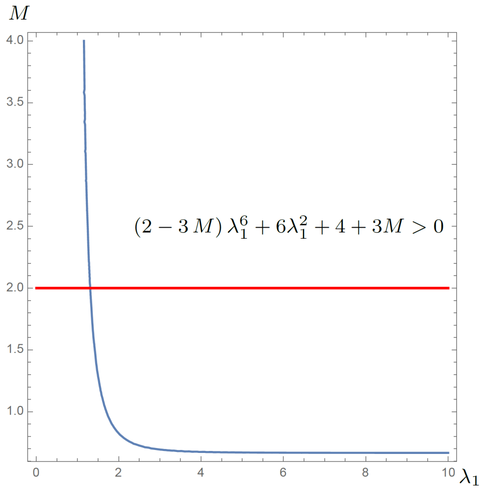

Figure 3: The plot of the pairs (λ_1,M) satisfying $(2-3M)\lambda_1^6+6\lambda_1^2+4+3M=0$. For fixed $M=2$, the unique solution is the intersection of the red line to the blue curve.

Surely, we have

$$s(x^2)=(-3M+2)x^6+6x^2+4+3M. \tag{4.13}$$

Therefore, $(-3M+2)x^6+6x^2+4+3M=0$ has a positive solution λ_1 if and only if $s(x)=0$ has a positive solution. But the function s is concave (see Fig. 4), since

$$s''(x)=6(2-3M)x<0, \quad \forall x>0, \quad \forall M>\frac{2}{3}, \tag{4.14}$$

and it attains its maximum in the stationary point, i.e. in the solution of the equation

$$s'(x)=0 \iff 6+(6-9M)x^2=0 \iff x=\frac{\sqrt{2}}{\sqrt{3M-2}}>0. \tag{4.15}$$

Note that

$$\begin{aligned} &s(0)=3M+4>0,\\ &s\left(\frac{\sqrt{2}}{\sqrt{3M-2}}\right)=3M+\frac{4}{\sqrt{3M/2-1}}+4>0,\\ &\lim_{x\to\infty} s(x)=-\infty. \end{aligned} \tag{4.16}$$

Thus, by the concavity, $s(x)$ remains positive at least until it reaches its maximum x_0 and, starting from x_0, $s(x)$ is strictly monotone decreasing. Since $s(x)$ is continuous and $\lim_{x\to\infty} s(x) \to -\infty$, there must be exactly one point $\widetilde{x}$, for which $s(\widetilde{x})=0$ by the intermediate value theorem and the strict monotonicity (starting from x_0), meaning that $\widetilde{x}$ is the unique solution to $s(x)=0$. $\square$

According to Theorem 3.2, strong monotonicity of the Biot stress-stretch relation for the Neo-Hooke-Ciarlet-Geymonat energy implies the positive semi-definiteness of the matrix $\mathrm{D}\widehat{T}$. Note however that, being not invertible and symmetric, the matrix $\mathrm{D}\widehat{T}$ is also not positive definite everywhere.

Moreover, as visualized for $M=1$ via numerical simulation in Fig. 5, the matrix $\mathrm{D}\widehat{T}$ is not positive semi-definite on $\mathbb{R}^3_+$ in general.

Proposition 4.2. *For the compressible Neo-Hooke-Ciarlet-Geymonat materials, the Biot stress-stretch relation is in general not monotone.*

Proof. Note that $\mathrm{D}\widehat{T}(\lambda_1,\lambda_1,\lambda_1)$ is a symmetric 3×3 matrix having the principal minors

$$\begin{aligned}
m_1^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)&:=\frac{(3M-2)\left(\lambda_1^6+1\right)/6+\lambda_1^2+1}{\lambda_1^2}>0, \quad M>\frac{2}{3},\\
m_2^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)&:=\frac{\left[3(3M-2)\lambda_1^6+6\lambda_1^2+4+3M\right]\left[(2-3M)\lambda_1^6+6\lambda_1^2+4+3M\right]}{36\lambda_1^4}, \qquad (4.17)\\
m_3^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)&:=\frac{\left[(2-3M)\lambda_1^6+3M+6\lambda_1^2+4\right]^2\left[5(3M-2)\lambda_1^6+3M+6\lambda_1^2+4\right]}{216\lambda_1^6}.
\end{aligned}$$

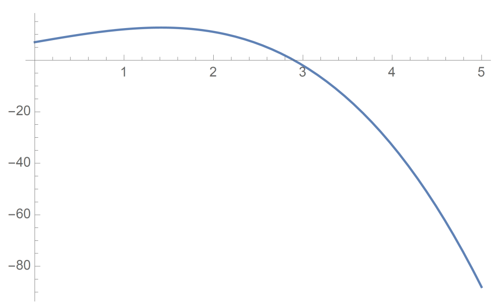

Figure 4: For $M=1$, the plot of $s:(0,\infty)\to\mathbb{R}$, $s(x)=(-3M+2)x^3+6x+3M+4$.

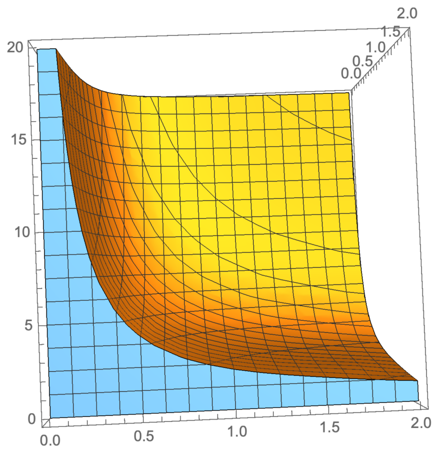

Figure 5: For $M=1$, the region plot of those $(\lambda_1,\lambda_2,\lambda_3)$ for which the matrix $\mathrm{D}\widehat{T}$ is positive definite.

The curve of those $(\lambda_1,M)\in\mathbb{R}_+\times(2/3,\infty)$ such that (4.11) is satisfied, divides the plane into two parts, one part at which

$$(2-3M)\lambda_1^6+6\lambda_1^2+4+3M>0$$

(above the blue curve) and the part at which

$$(2-3M)\lambda_1^6+6\lambda_1^2+4+3M<0$$

(below the blue curve). For each fixed $M>2/3$, see Fig. 3, for $\lambda_1<\lambda^*$, where λ^* corresponds to the intersection point of the red curve with the blue curve, the pairs (λ_1,M) are on the left-hand side of the blue, so

$$(2-3M)\lambda_1^6+6\lambda_1^2+4+3M<0,$$

while for $\lambda_1>\lambda^*$, the pairs (λ_1,M) are on the right-hand side of the blue, so

$$(2-3M)\lambda_1^6+6\lambda_1^2+4+3M>0.$$

Therefore, we have

$$\begin{aligned}
&m_1^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_1)>0, \quad m_2^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_1)>0, \quad m_3^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_1)>0, \quad \lambda_1<\lambda^*,\\
&m_1^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_1)>0, \quad m_2^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_1)=0, \quad m_3^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_1)=0, \quad \lambda_1=\lambda^*,\\
&m_1^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_1)>0, \quad m_2^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_1)<0, \quad m_3^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_1)>0, \quad \lambda_1>\lambda^*,
\end{aligned} \tag{4.18}$$

and, according to the Sylvester criterion, the proof is complete. □

5 Existence of the radial solution for general Neo-Hooke models

We have shown that the map $T_{\text{Biot}}:\text{Sym}^{++}(3)\to\text{Sym}(3)$ is, in general, not a diffeomorhpism. However, even if T_{Biot} is not surjective, in the construction of a homogeneous solution of Rivlin's cube problem, this does not immediately imply that the Eq. (2.8) does not have a solution. Moreover, since it is unclear yet whether T_{Biot} is injective, (2.8) could have more than one solution. Equally, after the system

$$T_{\text{Biot}}(U)=\alpha\cdot\mathbb{1} \iff T_i(\lambda_1,\lambda_2,\lambda_3)=\alpha, \quad i=1,2,3 \tag{5.1}$$

is solved, one may ask whether $\widehat{T}(\lambda_1,\lambda_2,\lambda_3)$ or $T_{\rm Biot}$ is locally strongly monotone in the solutions or if the homogeneous solutions are locally unique minimizers or energetically stable. Recall that the stability condition and global monotonicity were defined in Sections 3.2 and 3.3, while local strong monotonicity in $U\in{\rm Sym}^{++}(3)$ means that there exists $c_+>0$ such that for sufficiently small $\varepsilon>0$,

$$\langle T_{\rm Biot}(\widetilde{U})-T_{\rm Biot}(U),\widetilde{U}-U\rangle>c_+\|\widetilde{U}-U\|^2,\quad \forall\widetilde{U}\in{\rm Sym}^{++}(3) \tag{5.2}$$

such that $\|\widetilde{U}-U\|<\varepsilon$. We also note that (local) strict monotonicity implies the (local) uniqueness of the solution of (5.1), since otherwise, assuming that U_1 and U_2 are two different solutions,

$$\langle T_{\rm Biot}(U_1)-T_{\rm Biot}(U_2),U_1-U_2\rangle=0, \tag{5.3}$$

which contradicts the (local) strict monotonicity.

In this section, we consider the general models for the classical Neo-Hooke-type energies, i.e.

$$W_{\rm NH}(F)=\frac{\mu}{2}\langle C,\mathbb{1}\rangle+h(\det F)=\frac{\mu}{2}\|F\|^2+h(\det F)=\frac{\mu}{2}\|U\|^2+h(\det U). \tag{5.4}$$

The entire study is actually equivalent to the study of the one-parameter model described by the energy

$$W^M_{\rm NH}(F):=\frac{1}{\mu}W_{\rm NH}(F)=\frac{1}{2}\langle C,\mathbb{1}\rangle+\frac{1}{\mu}h(\det F)=\frac{1}{2}\|U\|^2+\frac{1}{\mu}h(\det U). \tag{5.5}$$

The corresponding first Piola-Kirchhoff stress tensor for this one parameter energy is given by

$$S_1^{\rm NH}=F+\frac{1}{\mu}h'(\det F)\cdot{\rm Cof}F, \tag{5.6}$$

and the Biot stress tensor is

$$T^{\rm NH}_{\rm Biot}(U)={\rm D}_U W_{\rm NH}(U)=R^\top S_1=U+\frac{1}{\mu}h'(\det U)\cdot\det U\cdot U^{-1}. \tag{5.7}$$

In order to have a stress free reference configuration, the function h has to satisfy $3/2+h'(1)/\mu=0$. Since $\mu>0$, we have $h'(1)<0$.

The first step in the study of the Rivlin cube problem is to check if a radial Biot stress tensor $T^{\rm NH}_{\rm Biot}=\alpha\,\mathbb{1}$ leads to a unique radial solution $U=\beta^+\,\mathbb{1}$ of the equation

$$T^{\rm NH}_{\rm Biot}(U)=\alpha\,\mathbb{1}. \tag{5.8}$$

Proposition 5.1. *For a hypperelastic material of the form* (5.4)*, if the Eq.* (5.8) *has a unique radial solution* $U=\beta^+\,\mathbb{1}$*,* $\beta^+>0$ *for every* $\alpha\in\mathbb{R}$*, then the convex function* h *satisfies*

$$\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)'\geq 0,\quad \forall x>0, \tag{5.9a}$$

$$\lim_{x\to 0}\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)=-\infty,\quad \lim_{x\to\infty}\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)=\infty. \tag{5.9b}$$

If the convex function h *satisfies*

$$\begin{aligned}&\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)'>0,\quad \forall x>0,\\ &\lim_{x\to 0}\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)=-\infty,\quad \lim_{x\to\infty}\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)=\infty,\end{aligned} \tag{5.10}$$

then the Eq. (5.8) *has a unique radial solution* $U=\beta^+\,\mathbb{1},\beta^+>0$ *for every* $\alpha\in\mathbb{R}$*.*

Proof. Eq. (5.8), after multiplication with U, reads

$$U^2+\frac{1}{\mu}h'(\det U)\cdot\det U\cdot\mathbb{1}=\alpha\cdot U. \tag{5.11}$$

This system has a radial solution $U=\beta^+\cdot\mathbb{1}$ if β^+ is a solution to the equation

$$\beta^+ +\frac{1}{\mu}h'\left((\beta^+)^3\right)(\beta^+)^2=\alpha, \tag{5.12}$$

or with the substitution $x=(\beta^+)^3$, if there is a unique positive solution x of the equation

$$\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}=\alpha. \tag{5.13}$$

There exists at least one solution x of the Eq. (5.13) if and only if for each $\alpha\in\mathbb{R}$ the function $x\mapsto\sqrt[3]{x}+h'(x)\sqrt[3]{x^2}/\mu$ is not bounded on $(0,\infty)$. Otherwise, there exist values of α, smaller or larger than the lower bound or upper bound, respectively, for which the function $x\mapsto\sqrt[3]{x}+h'(x)\sqrt[3]{x^2}/\mu$ never reach these values of α. On the other hand, if the function is unbounded, then if the function $x\mapsto\sqrt[3]{x}+h'(x)\sqrt[3]{x^2}/\mu$ were not monotone, then the Eq. (5.13) could have more than one solution for some $\alpha\in\mathbb{R}$. In conclusion, for a given $\alpha\in\mathbb{R}$, if the Eq. (5.8) has a unique solution then the convex function h has one of the following properties:

$$\begin{aligned}&\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)'\geq 0, \quad \forall x>0,\\ &\lim_{x\to 0}\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)=-\infty, \quad \lim_{x\to\infty}\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)=\infty,\end{aligned} \tag{5.14}$$

or

$$\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)'\leq 0, \quad \forall x>0, \tag{5.15a}$$

$$\lim_{x\to 0}\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)=\infty, \quad \lim_{x\to\infty}\left(\sqrt[3]{x}+\frac{1}{\mu}h'(x)\sqrt[3]{x^2}\right)=-\infty. \tag{5.15b}$$

Since h is convex, h' is monotone increasing. Hence,

$$\begin{aligned}&h'(x)>h'(1), &&\forall x>1,\\ &h'(x)<h'(1)<0, &&\forall x<1.\end{aligned} \tag{5.16}$$

The conditions (5.15b) is therefore not admissible, since (5.15b) implies that

$$\lim_{x\to 0}\left(h'(x)\sqrt[3]{x^2}\right)=\infty,$$

which is not possible (since (5.16) yields $h'(x)\sqrt[3]{x^2}<0$ for all $x<1$). Hence, it remains that if the system (5.8) has a unique radial solution, then h has to satisfy the conditions (5.10).

Finally, note that, the last part of the conclusions, the uniqueness of β^+ is implied by the strict monotonicity of the mapping $x\mapsto(\sqrt[3]{x}+h'(x)\sqrt[3]{x^2}/\mu)$ and by the limit conditions. □

6 Bifurcation in Rivlin's cube problem for the compressible Neo-Hooke-Ciarlet-Geymonat model

Let us now consider the Neo-Hooke-Ciarlet-Geymonat model, i.e. the Neo-Hooke model for which $h(x)/\mu$ is given by the function

$$h^M_{\rm CG}(x)=-\log x+\left(\frac{M}{4}-\frac{1}{6}\right)(x^2-2\log x-1). \tag{6.1}$$

For the Neo-Hooke-Ciarlet-Geymonat model, Rivlin's cube problem amounts to finding the solutions of the nonlinear algebraic system

$$T_1\equiv\lambda_1-\frac{1}{\lambda_1}+\left(\frac{M}{2}-\frac{1}{3}\right)\left(\lambda_1\lambda_2^2\lambda_3^2-\frac{1}{\lambda_1}\right)=\alpha, \tag{6.2a}$$

$$T_2 \equiv \lambda_2 - \frac{1}{\lambda_2} + \left(\frac{M}{2} - \frac{1}{3}\right)\left(\lambda_1^2\lambda_2\lambda_3^2 - \frac{1}{\lambda_2}\right) = \alpha, \tag{6.2b}$$

$$T_3 \equiv \lambda_3 - \frac{1}{\lambda_3} + \left(\frac{M}{2} - \frac{1}{3}\right)\left(\lambda_1^2\lambda_2^2\lambda_3 - \frac{1}{\lambda_3}\right) = \alpha. \tag{6.2c}$$

6.1 Radial solution: Three equal stretches

When we are looking for a radial solution $(\lambda_1,\lambda_2,\lambda_3) = (\beta^+,\beta^+,\beta^+)$ of the system (6.2), we are looking for a solution of the equation

$$T_1(\beta^+,\beta^+,\beta^+) \equiv \frac{(3M-2)(\beta^+)^6 + 6(\beta^+)^2 - 4 - 3M}{6\beta^+} = \alpha. \tag{6.3}$$

As shown, such a solution exists, and its uniqueness is equivalent to the conditions on $h_{\rm CG}$ from Proposition 5.1. For the Neo-Hooke-Ciarlet-Geymonat model, condition (5.9) is

$$\frac{(3M-2)(5x^2+7)}{216x\sqrt[3]{x}} + \frac{1}{3\sqrt[3]{x^2}} > 0, \quad \forall x > 0, \tag{6.4}$$

which is clearly satisfied for $M = (\lambda + 2\mu/3)/\mu > 2/3$. The conditions (5.9b) are also satisfied, since

$$\begin{aligned} &\lim_{x\to 0}\left(\frac{(3M-2)\sqrt[3]{x^2}\,(x^2-7)}{72x} + \sqrt[3]{x}\right) = -\infty, \\ &\lim_{x\to \infty}\left(\frac{(3M-2)\sqrt[3]{x^2}\,(x^2-7)}{72x} + \sqrt[3]{x}\right) = \infty. \end{aligned} \tag{6.5}$$

Therefore, the corresponding radial solutions are unique.

Proposition 6.1. *For the compressible Neo-Hooke-Ciarlet-Geymonat material, for all* $\alpha \in \mathbb{R}$*, the constitutive equation* $T_{\rm Biot}(U) = \alpha\,\mathbb{1}$ *has a unique radial solution* $U = \beta^+ \cdot \mathbb{1}_3$*,* $\beta^+ > 0$*. Moreover, since the mapping*

$$\begin{aligned} &f_{\rm Biot}: (0,\infty) \;\to\; \mathbb{R}, \\ &f_{\rm Biot}(\beta^+) := T_1(\beta^+,\beta^+,\beta^+) = \frac{(3M-2)(\beta^+)^6 + 6(\beta^+)^2 - 4 - 3M}{6\beta^+} \end{aligned} \tag{6.6}$$

is strictly monotone increasing, continuous and surjective (see Fig. 6), the solution $\beta^+ = \beta^+(\alpha)$ *is a monotone increasing function (see Fig. 7).*

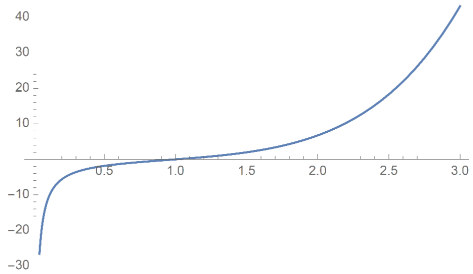

Figure 6: The plot of $\beta^+ \mapsto f_{\text{Biot}}(\beta^+) = T_1(\beta^+,\beta^+,\beta^+)$.

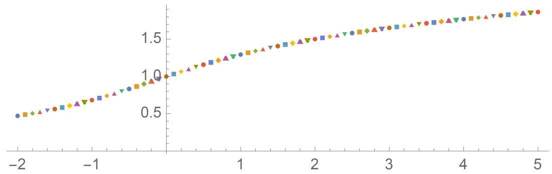

Figure 7: The solution β^+ of $T_1(\beta^+,\beta^+,\beta^+)=\alpha$ depends strictly monotone on α.

However, when the bifurcation problem is studied in the Rivlin's cube problem, we are interested to study if for all $\alpha>0$ all radial solutions $U=\beta^+\cdot\mathbb{1}$ of the equation

$$T^{\text{NH}}_{\text{Biot}}(U)=\alpha\,\mathbb{1} \tag{6.7}$$

are locally unique in the general classes of all possible solutions $U\in\text{Sym}^{++}(3)$ (possibly non-radial), see Table 1 for a summary of the constitutive conditions used in this paper.

Table 1: A summary of the constitutive conditions used in this paper. Here, $\epsilon_{ij}=1$ if $\{i,j\}=\{1,2\}$ or $\{2,3\}$ or $\{3,1\}$, $\epsilon_{ij}=-1$ otherwise.

In terms of	the principal Biot stresses $\widehat{T}=(T_1,T_2,T_3)^\top$	the energy expressed in the principal stretches $W(F)=g(\lambda_1,\lambda_2,\lambda_3)$
invertibility of T_{Biot}	$\mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)$ is invertible for any $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3_+$ and $\lVert\widehat{T}(\lambda_1,\lambda_2,\lambda_3)\rVert_{\mathbb{R}^3}\to\infty$ as $\lVert(\log\lambda_1,\log\lambda_2,\log\lambda_3)\rVert_{\mathbb{R}^3}\to\infty$	$\mathrm{D}^2g(\lambda_1,\lambda_2,\lambda_3)$ is invertible for any $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3_+$ and $\lVert\mathrm{D}g(\lambda_1,\lambda_2,\lambda_3)\rVert_{\mathbb{R}^3}\to\infty$ as $\lVert(\log\lambda_1,\log\lambda_2,\log\lambda_3)\rVert_{\mathbb{R}^3}\to\infty$
Hilbert-monotonicity of T_{Biot}	$\mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)\in\text{Sym}^+(3)$	$\mathrm{D}^2g(\lambda_1,\lambda_2,\lambda_3)\in\text{Sym}^+(3)$
strong-monotonicity of $\widehat{T}$	$\mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)\in\text{Sym}^{++}(3)$	$\mathrm{D}^2g(\lambda_1,\lambda_2,\lambda_3)\in\text{Sym}^{++}(3)$
energetic stability	$\dfrac{T_i-\epsilon_{ij}T_j}{\lambda_i-\epsilon_{ij}\lambda_j}\geq 0$, $i\neq j$ no sum and $\mathrm{D}\widehat{T}(\lambda_1,\lambda_2,\lambda_3)\in\text{Sym}^+(3)$	$\dfrac{\partial g/\partial\lambda_i-\epsilon_{ij}\partial g/\partial\lambda_j}{\lambda_i-\epsilon_{ij}\lambda_j}\geq 0$, $i\neq j$ no sum and $\mathrm{D}^2g(\lambda_1,\lambda_2,\lambda_3)\in\text{Sym}^+(3)$

Note that we are not interested in having a unique solution of (6.7), but a locally unique solution. This is because we have to study whether the solution may continuously (in the sense of the continuity of the map $\alpha \mapsto U(\alpha)$) depart from a radial one to a non-radial one and vice-versa. This is only possible in those points in which the mapping $U \mapsto T_{\rm Biot}(U)$ is not invertible, i.e. using Theorem 3.1, in those points $(\lambda_1^*,\lambda_2^*,\lambda_3^*)$ where

$$\det {\rm D}\widehat{T}(\lambda_1^*,\lambda_2^*,\lambda_3^*)=0. \tag{6.8}$$

Specifically, we are thus interested in the existence of a radial $U=\lambda_1 \mathbb{1}$ such that

$$\det {\rm D}\widehat{T}(\lambda_1,\lambda_1,\lambda_1)=0, \tag{6.9}$$

i.e. whether the map $U \mapsto T_{\rm Biot}(U)$ loses local differentiable invertibility in a radial U. Indeed, for each material given by $M>2/3$ we have that

$$\begin{aligned}&\det {\rm D}\widehat{T}(\lambda_1,\lambda_1,\lambda_1)\\ =&\frac{\left[(2-3M)\lambda_1^6+6\lambda_1^2+4+3M\right]^2\left[5(3M-2)\lambda_1^6+6\lambda_1^2+4+3M\right]}{216\lambda_1^6},\end{aligned} \tag{6.10}$$

and therefore, $U \mapsto T_{\rm Biot}(U)$ loses the local invertibility in $U=\lambda^* \mathbb{1}$, where λ^* is the unique solution (see the proof of Proposition 4.1) of the equation

$$(2-3M)\lambda_1^6+6\lambda_1^2+4+3M=0. \tag{6.11}$$

Since for $M>2/3$ the above equation has a unique positive solution, see the proof of Proposition 4.1 and Fig. 3 (for fixed M, the unique solution is the intersection of the red line with the blue curve), we argue that the bifurcation occurs for all admissible constitutive parameters in only one radial solution.

We recall that, from the proof of Proposition 4.2, we have

$$\begin{aligned}
&m_1^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)>0, \quad m_2^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)>0, \quad m_3^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)>0, \quad \lambda_1<\lambda^*,\\
&m_1^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)>0, \quad m_2^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)=0, \quad m_3^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)=0, \quad \lambda_1=\lambda^*,\\
&m_1^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)>0, \quad m_2^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)<0, \quad m_3^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)>0, \quad \lambda_1>\lambda^*,
\end{aligned} \tag{6.12}$$

where $m_1^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1), m_2^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)$ and $m_1^{\rm Biot}(\lambda_1,\lambda_1,\lambda_1)$ are the principal minors of ${\rm D}\widehat{T}(\lambda_1,\lambda_1,\lambda_1)$.

Hence, even if the map $\alpha \mapsto \beta(\alpha)$ giving the solution of $T_{\rm Biot}(\beta \mathbb{1})=\alpha \mathbb{1}$ is strictly monotone increasing, the relation $T_{\rm Biot}=T_{\rm Biot}(U)$ could be locally strictly monotone only at those radial $U=\lambda_1 \mathbb{1}$ for which $\lambda_1<\lambda^*$, and it loses its strict monotonicity on those radial $U=\lambda_1 \mathbb{1}$ for which $\lambda_1>\lambda^*$, see Figs. 8 and 9. Moreover, $\widehat{T}$

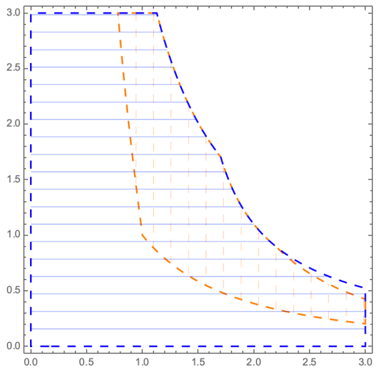

Figure 8: For $M=1$, the strong monotonicity region in the points $(\lambda_1,\lambda_1,\lambda_2)$ (the blue region) versus the energetic stability region in the points $(\lambda_1,\lambda_1,\lambda_2)$ (the orange region).

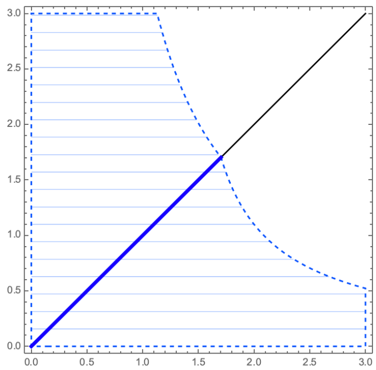

Figure 9: The strong monotonicity is satisfied on the radial solutions until the bifurcation point (on the blue curve), i.e. $0<\lambda_1\leq\lambda^*$.

is strongly monotone only for $\lambda_1<\lambda^*$. This is unphysical, since for purely radial deformations, the Biot stress should clearly increase with the strain; therefore, for $U=\lambda_1^*\mathbb{1}$, $\lambda_1\geq\lambda^*$, the radial solution should not be considered physically admissible anymore. In other words, the cube cannot remain a cube by increasing its length above λ^* and at the same time keeping the strict monotonicity of the $T_{\rm Biot}=T_{\rm Biot}(U)$ relation.

Regarding the energetic stability of the radial solution, we note that the stability conditions (3.13) against arbitrary perturbations for radial solutions, by letting $\lambda_i\to\lambda$, read as

$$\left(\frac{\partial^2 g}{\partial\lambda_1^2}-\frac{\partial^2 g}{\partial\lambda_1\partial\lambda_2}\right)\bigg|_{\lambda_i=\lambda}\geq 0,\quad \left(\frac{\partial g/\partial\lambda_2+\partial g/\partial\lambda_1}{2\lambda_1}\right)\bigg|_{\lambda_i=\lambda}\geq 0. \tag{6.13}$$

In addition to these conditions, energetic stability requires to check the positive semi-definiteness of the Hessian matrix evaluated in the radial solution $\mathrm{D}^2 g=(\partial^2 g/(\partial\lambda_i\partial\lambda_j))|_{\lambda_i=\lambda}$, too.

The first inequality is equivalent to

$$\frac{(2-3M)\lambda_1^6+6\lambda_1^2+3M+4}{6\lambda_1^2}\geq 0, \tag{6.14}$$

while the second one is equivalent to

$$\frac{(3M-2)\lambda_1^6+6\lambda_1^2-4-3M}{3\lambda_1^2}\geq 0. \tag{6.15}$$

Note that the Hessian matrix $\mathrm{D}^2 g$ is actually $\mathrm{D}\widehat{T}(\lambda_1,\lambda_1,\lambda_1)$, and therefore the last condition for stability, i.e. the positive semi-definiteness of $\mathrm{D}^2 g$, is implied by the strict monotonicity of $\widehat{T}$. Moreover, the first inequality required by the stability criterion is redundant, and so it follows from the local positive definiteness of $\mathrm{D}\widehat{T}(\lambda_1,\lambda_1,\lambda_1)$, see the expression of $m_1^{\mathrm{Biot}}(\lambda_1,\lambda_1,\lambda_1)$. In conclusion, the stability of the radial solutions is implied by the strong monotonicity of $\widehat{T}$ in the radial solutions, which holds true only until the radial solution reaches the bifurcation point, i.e. for $\lambda_i \le \lambda^*, i=1,2,3$, together with

$$(3M-2)\lambda_1^6+6\lambda_1^2-4-3M \ge 0. \tag{6.16}$$

Since,

$$T_1(\lambda_1,\lambda_1,\lambda_1)=\frac{(3M-2)\lambda_1^6+6\lambda_1^2-4-3M}{6\lambda_1},$$

(6.16) is possible only when $T_{\mathrm{Biot}}(\mathrm{diag}(\lambda_1,\lambda_1,\lambda_1))=\alpha\mathbb{1}$ with $\alpha\ge 0$. Note again that $\lambda_1 \mapsto T_1(\mathrm{diag}(\lambda_1,\lambda_1,\lambda_1)), \lambda_1>0$ is strictly monotone increasing, continuous and surjective, $\lim_{\lambda_1\to\infty} T_1(\lambda_1,\lambda_1,\lambda_1)=\infty, \lim_{\lambda_1\to-\infty} T_1(\lambda_1,\lambda_1,\lambda_1)=-\infty$ and $T_1(1,1,1)=0$. Hence, inequality (6.16) holds true only for $\lambda_1\ge 1$, i.e. when the radial solution is a uniform extension.

Hence, the radial solution is stable, see Fig. 10, only for

$$1 \le \lambda_1 \le \lambda^*, \tag{6.17}$$

which lets us conclude that the stability criteria for the radial solutions are more restrictive than the monotonicity criteria.

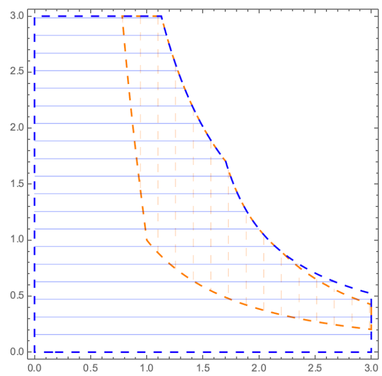

Figure 10: Contrary to strong monotonicity, the energetic stability is satisfied on the radial solutions only starting with 1 and until the bifurcation point (on the blue curve), i.e. $1\le\lambda_1\le\lambda^*$.

6.2 Two equal principal stretches $(\lambda_1=\lambda_2\neq\lambda_3)$

In this subsection, we find the solutions of the form

$$U=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{pmatrix},\quad \lambda_1\neq\lambda_2$$

of the equation $T_{\rm Biot}(U)=\alpha\,\mathbb{1}$. We show that in the neighbourhood of the unique radial solution $U=\lambda_1^*\,\mathbb{1}$ for which $\det \mathrm{D}\widehat{T}(\lambda_1^*,\lambda_1^*,\lambda_1^*)$ is not invertible, i.e. in a neighbourhood of $\alpha^*=T_1(\lambda_1^*,\lambda_1^*,\lambda_1^*)$, the equation $T_{\rm Biot}(U)=\alpha_\epsilon\,\mathbb{1}$ admits a solution

$$U_\epsilon=\begin{pmatrix}\lambda_1^\varepsilon & 0 & 0\\ 0 & \lambda_1^\varepsilon & 0\\ 0 & 0 & \lambda_2^\varepsilon\end{pmatrix},\quad \lambda_1^\varepsilon\neq\lambda_2^\varepsilon,$$

which tends to the radial solution

$$U^*=\begin{pmatrix}\lambda_1^* & 0 & 0\\ 0 & \lambda_1^* & 0\\ 0 & 0 & \lambda_1^*\end{pmatrix},$$

when α_ε goes to α^*. Hence, we have to solve the system

$$\begin{aligned}&\frac{(3M-2)\left(\lambda_1^4\lambda_2^2-1\right)/6+\lambda_1^2-1}{\lambda_1}=\alpha,\\ &\frac{1}{6}(3M-2)\lambda_1^4\lambda_2-\frac{3M+4}{6\lambda_2}+\lambda_2=\alpha,\end{aligned}\tag{6.18}$$

since

$$\begin{aligned}&T_1(\beta,\beta,\gamma)=T_2(\beta,\beta,\gamma)=\frac{(3M-2)\left(\beta^4\gamma^2-1\right)/6+\beta^2-1}{\beta},\\ &T_3(\beta,\beta,\gamma)=\frac{1}{6}(3M-2)\beta^4\gamma-\frac{3M+4}{6\gamma}+\gamma,\end{aligned}$$

or equivalently to solve the system

$$\begin{aligned}&T_1(\lambda_1,\lambda_1,\lambda_2)-T_3(\lambda_1,\lambda_1,\lambda_2)\\ &\equiv-\frac{(\lambda_1-\lambda_2)\left((3M-2)\lambda_1^4\lambda_2^2-3M-6\lambda_1\lambda_2-4\right)}{6\lambda_1\lambda_2}=0,\\ &\frac{(3M-2)\left(\lambda_1^4\lambda_2^2-1\right)/6+\lambda_1^2-1}{\lambda_1}=\alpha.\end{aligned}\tag{6.19}$$

Hence, $\lambda_1 = \lambda_2$, i.e. yielding the radial solution obtained in the previous section, or

$$\lambda_2 = \frac{\sqrt{(9M^2+6M-8)\lambda_1^4+9\lambda_1^2}+3\lambda_1}{(3M-2)\lambda_1^4} \tag{6.20}$$

with λ_1 being a solution of

$$\frac{\sqrt{(9M^2+6M-8)\lambda_1^4+9\lambda_1^2}+(3M-2)\lambda_1^5+3\lambda_1}{(3M-2)\lambda_1^4} = \alpha. \tag{6.21}$$

For any $M > 2/3$, the function

$$\ell : (0,\infty) \to (0,\infty), \quad \ell(\lambda_1) = \frac{\sqrt{(9M^2+6M-8)\lambda_1^4+9\lambda_1^2}+(3M-2)\lambda_1^5+3\lambda_1}{(3M-2)\lambda_1^4} \tag{6.22}$$

is convex. Moreover, we have

$$\lim_{\lambda_1 \to 0} \ell(\lambda_1) = \infty = \lim_{\lambda_1 \to \infty} \ell(\lambda_1). \tag{6.23}$$

Thus,

- For $\alpha < \min_{\lambda_1 > 0} \ell(\lambda_1)$ the Eq. (6.21) has no solutions. Therefore, only the radial solution which always exists is a solution of the equation $T_{\rm Biot}(U) = \alpha \mathbb{1}$.
- For $\alpha = \min_{\lambda_1 > 0} \ell(\lambda_1)$ the Eq. (6.21) has one solution. Hence, the equation $T_{\rm Biot}(U) = \alpha \mathbb{1}$ has two solutions: one radial and another with two equal eigenvalues.
- For all $\alpha > \min_{\lambda_1 > 0} \ell(\lambda_1)$ the Eq. (6.21) has two different admissible solutions, which lead to two non-radial admissible solutions of the equation $T_{\rm Biot}(U) = \alpha \mathbb{1}$. Besides these two solutions we have the already found radial solution, too.

More about the behaviour of these solutions may be observed from Fig. 11, using $M = 1$. For values of α below the green line, there exists only the radial solution. Between the green and the red lines there exist the radial solution and two other non-radial solutions with two equal eigenvalues. By approaching the red line, one non-radial solutions goes to the radial solution situated at the intersection point of the orange curve with the blue curve, while the other non-radial solution tends into the other direction of the blue curve and will never be in the neighbourhood of a radial solution. For values of α above the red line, there are again three different solutions.

It is easy to find that no bifurcation occurs in compression, since for $\alpha < 0$ there exists only the radial solution. Thus, at the radial solution given by the intersection point of the blue and orange curve, the relation $T_{\rm Biot} = T_{\rm Biot}(U)$ is not locally invertible, since the radial solution is not locally unique.

In conclusion, a bifurcation point λ^* is a solution of (6.11). In Fig. 14 we plot the path of the point (λ_1, λ_2) as a function of $\alpha \in [0,5]$ with a step size of 0.1, by solving numerically the Eq. (6.21) for $M = 1$, but the analysis is completely the same for any other value of M.

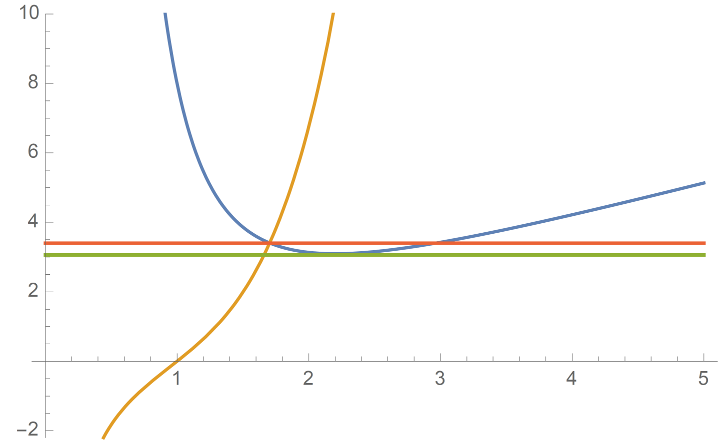

Figure 11: The plot of $\beta \mapsto \ell(\beta)$ (blue curve) and plot of $\beta \mapsto f_{\rm Biot}(\beta)$ (orange curve) for $M = 1$.

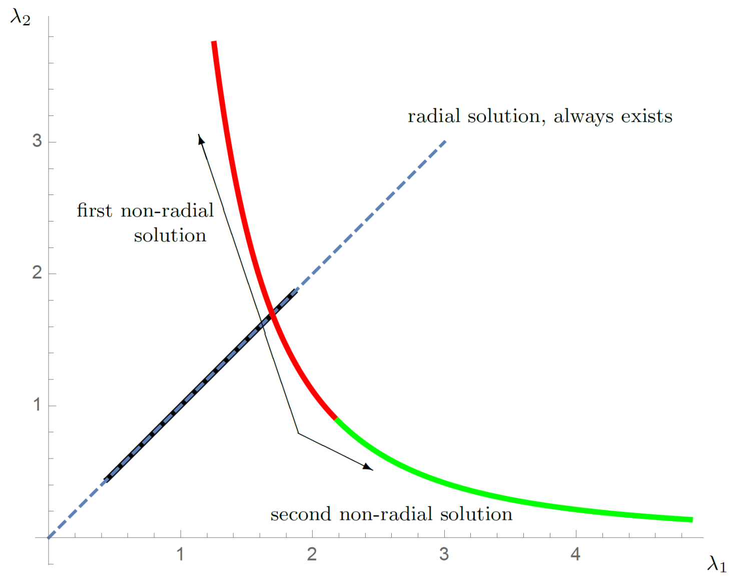

Figure 12: For $M = 1$, radial and non-radial solutions, bifurcation.

However, for one branch of non-radial solutions (green curve) there is no value of the Biot-stress magnitude for which the cube may continuously switch from the radial solution to a non-radial solution, while for the other branch of non-radial solutions (red curve) there is a unique value of the Biot-stress magnitude for which the cube may continuously switch from the radial solution to a non-radial solution, see Figs. 13 and 14.

Regarding the strong monotonicity of the non-radial solutions, we remark that while the principal minor

$$m_1^{\rm Biot}(\lambda_1,\lambda_1,\lambda_2):=\left(\frac{M}{4}-\frac{1}{6}\right)\left(2\lambda_1^2\lambda_2^2+\frac{2}{\lambda_1^2}\right)+\frac{1}{\lambda_1^2}+1>0 \tag{6.24}$$

of $\mathrm{D}\widehat{T}(\lambda_1,\lambda_1,\lambda_2)$ is positive (not only on this curve), the second principal minor

$$\begin{aligned} &m_2^{\rm Biot}(\lambda_1,\lambda_1,\lambda_2)\\ :=&-\frac{\left((3M-2)\lambda_1^4\lambda_2^2-3M-6\lambda_1^2-4\right)\left(3(3M-2)\lambda_1^4\lambda_2^2+3M+6\lambda_1^2+4\right)}{36\lambda_1^4}, \end{aligned} \tag{6.25}$$

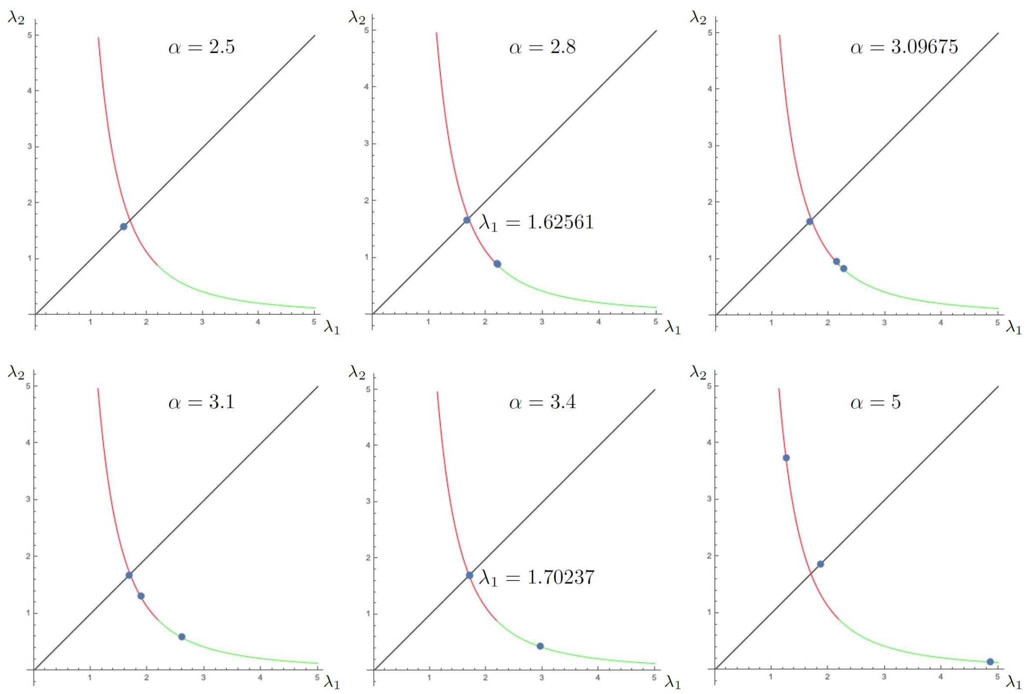

Figure 13: For $M=1$, the solutions of the equation $T_i(\lambda_1,\lambda_1,\lambda_2)=\alpha, i=1,2,3$ for a sequence of increasing values of α.

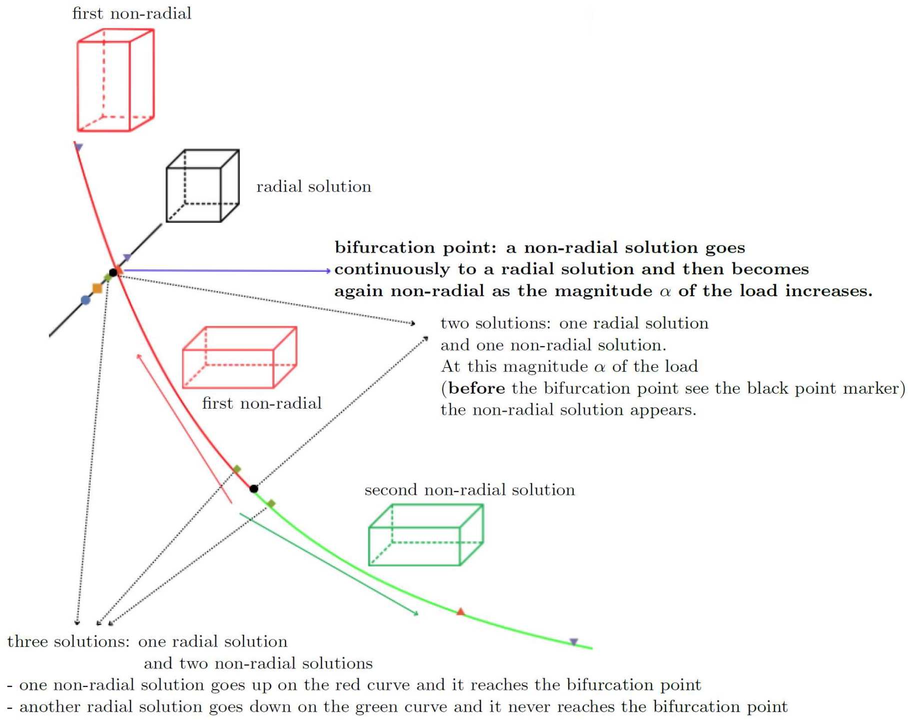

Figure 14: For $M=1$, a zoom commented picture of radial and non-radial solutions, bifurcation.

is strictly positive only on those points on this curve for which

$$(2-3M)\lambda_1^4\lambda_2^2+3M+6\lambda_1^2+4>0, \tag{6.26}$$

i.e. for λ_1 satisfying

$$\begin{aligned} &-\frac{6\left(-\sqrt{(9M^2+6M-8)\lambda_1^2+9}+(3M-2)\lambda_1^4-3\right)}{(3M-2)\lambda_1^2}>0 \\ \Longleftrightarrow\ &\left(\lambda_1^4>\frac{3}{3M-2} \quad \text{and} \quad -4-3M-6\lambda_1^2+(3M-2)\lambda_1^6>0\right). \end{aligned} \tag{6.27}$$

In fact, each λ_1 such that $4-3M-6\lambda_1^2+(3M-2)\lambda_1^6>0$ satisfies $\lambda_1^4>3/(3M-2)$, too. Indeed, if $\lambda_1^4\geq 3/(3M-2)$ then

$$-4-3M-6\lambda_1^2+(3M-2)\lambda_1^6\leq -4-3M-3\lambda_1^2<0.$$

Therefore, a necessary condition for the strong monotonicity of a non-radial solution is $\lambda_1>\lambda^*$. In Figs. 14 and 15, this means the part of the red curve below the $\lambda_1=\lambda_2$ curve and the entire green curve. The analytic study of the sign of the third principal minor $m_3^{\rm Biot}(\lambda_1,\lambda_1,\lambda_2)$ of $\mathrm{D}\widehat{T}(\lambda_1,\lambda_1,\lambda_2)$ is more complicated. However, the numerical testing has shown that on the entire red curve the strong monotonicity of the principal Biot stresses vector is lost, while on the green curve the monotonicity holds true, see Fig. 16.

Numerical computations show that, before and after the bifurcation point the values of the internal energy density $W_{\rm CG}(F)$ as well as the absolute value of the total energy (2.5) are smaller on the radial solutions, in comparison to the non-radial solutions, while this is not true for the total energy (2.5), even before the bifurcation. Note that the total energy is positive for contraction and negative for extension.

In the following we discuss the stability of the non-radial solutions, the stability of the radial solutions being already discussed in the previous subsection. The stability conditions (3.15) are equivalent to

$$\begin{aligned}
&\frac{(2-3M)\lambda_1^4\lambda_2^2+3M+6\lambda_1^2+4}{6\lambda_1^2}\geq 0, && \frac{(2-3M)\lambda_1^4\lambda_2^2+3M+6\lambda_1\lambda_2+4}{6\lambda_1\lambda_2}\geq 0,\\
&\frac{(3M-2)\lambda_1^4\lambda_2^2-3M+6\lambda_1^2-4}{6\lambda_1^2}\geq 0, && \frac{1}{6}\left((3M-2)\lambda_1^3\lambda_2-\frac{3M+4}{\lambda_1\lambda_2}+6\right)\geq 0.
\end{aligned} \tag{6.28}$$

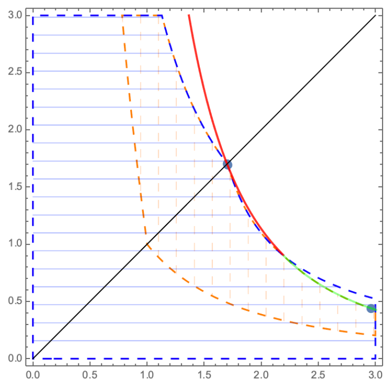

Figure 15: For $M=1$, the strong monotonicity region in the points $(\lambda_1,\lambda_1,\lambda_2)$ (the blue region) versus the energetic stability region in the points $(\lambda_1,\lambda_1,\lambda_2)$ (the orange region) together with the radial solutions (the black curve) and the non-radial solutions (the red curve and the green curve). Only one branch (the green curve) of non-radial solutions belongs to the strong monotonicity domain. The same branch (the green curve) belongs to the energetic stability domain, too.

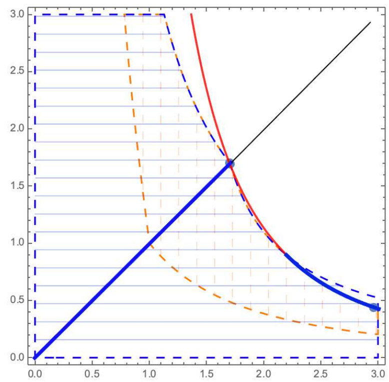

Figure 16: The strong monotonicity is satisfied on the radial solutions until the bifurcation point is reached, while after the value of α for which bifurcation is present, one branch of the radial solutions preserves the strong monotonicity, while the radial solutions and the other branch (red curve) do not satisfy the strong monotonicity conditions. In this figure, the strong monotonicity is satisfied on the blue curves.

The first two are equivalent to the positivity of $m_2^{\text{Biot}}(\lambda_1,\lambda_1,\lambda_2)$, so it implies $\lambda_1 \geq \lambda^*$ while the third implies $\alpha \geq 0$ which is always satisfied since the non-radial solution is present only in extension. For $\lambda_1 \geq \lambda^*$ it follows that the fourth inequality is satisfied, too.

The study of the positive semi-definiteness of $\mathrm{D}^2 g$ is similar to the study of the strong monotonicity of $\widehat{T}$ on the non-radial solution, which has already been treated above.

Summarising, the energetic stability of the non-radial solutions is equivalent to the monotonicity of the map $U \mapsto T_{\text{Biot}}(U)$ in these points. Therefore, the radial solutions are stable if and only if $1 \geq \lambda_1 \geq \lambda^*$, while the non-radial solutions are energetic stable only on the green branch of the non-radial solution. The energetic stable solutions are given in Fig. 17 by the orange curve.

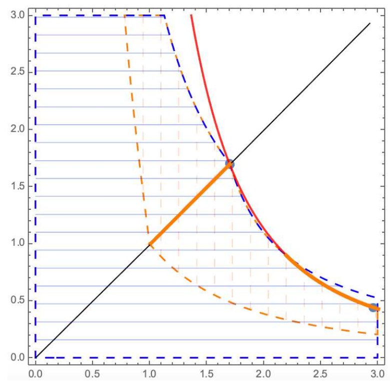

Figure 17: The energetic stability is satisfied on the radial solutions only starting with $\lambda_1 = 1$ and until the bifurcation point is attended. However, for non-radial solutions the situation is similar to the strong monotonicity, i.e. the stability is satisfied only on one branch of the radial solutions. In this figure, the energetic stability is satisfied on the orange curves.

6.3 Unequal principal stretches $(\lambda_i \neq \lambda_j,\, i \neq j)$

Using the expressions (4.6) of the principal Biot stresses we find that the general solution $(\lambda_1,\lambda_2,\lambda_3)$ of the equation $T_{\text{Biot}}(U) = \alpha \mathbb{1}$ is described by the following system:

$$
\begin{aligned}
&-\frac{(\lambda_1-\lambda_3)\big((3M-2)\lambda_1^2\lambda_2^2\lambda_3^2-3M-6\lambda_1\lambda_3-4\big)}{6\lambda_1\lambda_3}=0,\\
&-\frac{(\lambda_2-\lambda_3)\big((3M-2)\lambda_1^2\lambda_2^2\lambda_3^2-3M-6\lambda_2\lambda_3-4\big)}{6\lambda_2\lambda_3}=0,\\
&\frac{1}{6}(3M-2)\lambda_1^2\lambda_2^2\lambda_3-\frac{3M+4}{6\lambda_3}+\lambda_3=\alpha.
\end{aligned}
\tag{6.29}
$$

If $\lambda_1=\lambda_3$ or $\lambda_2=\lambda_3$, then we are in the situation of the previous section. If $\lambda_1\neq\lambda_2\neq\lambda_3\neq\lambda_1$ then

$$\begin{aligned}&(3M-2)\lambda_1^2\lambda_2^2\lambda_3^2-3M-6\lambda_1\lambda_3-4=0,\\&(3M-2)\lambda_1^2\lambda_2^2\lambda_3^2-3M-6\lambda_2\lambda_3-4=0,\end{aligned} \tag{6.30}$$

which implies that $\lambda_1=\lambda_2$. So the entire discussion reduces to the situation when two singular values are equal, a conclusion which may be observed from the numerical simulation given in Fig. 18,

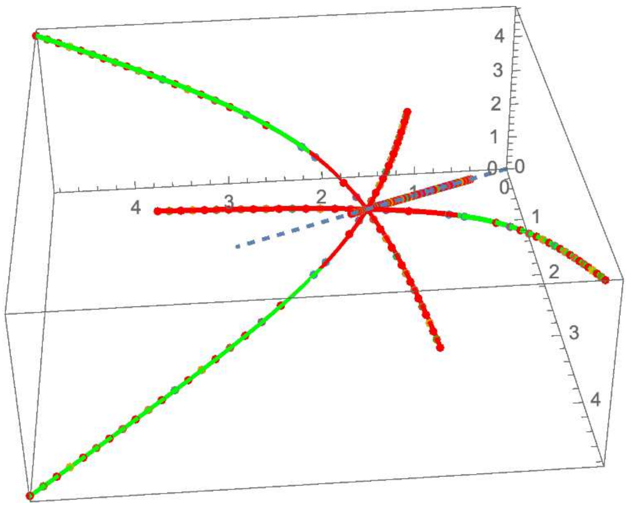

Figure 18: The numerical simulation for $M=1$ of the solutions $(\lambda_1,\lambda_2,\lambda_3)$ of the equation $T_{\rm Biot}(U)=\alpha\mathbb{1}$, $\alpha\in[-2,5]$.

7 Conclusion

In this study, we investigated the invertibility and monotonicity of stress-strain relations, specifically focusing on the Biot stress tensor-right stretch tensor relation and Rivlin's cube problem. Our primary objective was to determine the conditions under which a unique radial solution exists for Neo-Hooke type materials, where the cube remains a cube under any magnitude of radial stress.

We established that the function $h\equiv h_{CG}$ defining the Ciarlet-Geymonat energies meets the necessary and sufficient properties for ensuring the existence of a unique radial solution. For the Neo-Hooke-Ciarlet-Geymonat model, we identified both radial and non-radial solutions. In the extension case, non-radial solutions arise, transforming the cube into a parallelepiped, while in compression or below a critical force magnitude $\alpha^\flat$, such solutions do not exist. Our analysis revealed that radial solutions maintain local monotonicity up to a critical value α^*, beyond which bifurcation occurs and monotonicity is lost. This critical value corresponds to the point where invertibility is lost in radial solutions, in terms of principal Biot stresses and principal stretches. For force magnitudes starting from $\alpha^* \geq \alpha^\flat$ (below the bifurcation threshold), we identified two classes of non-radial solutions, both appearing discontinuously at α^* and then depending continuously on the force intensity. One class of non-radial solutions approaches the bifurcation branch, while the other set of non-radial solutions diverges from it. Numerical tests indicated that the first class of non-radial solutions fails to ensure strong monotonicity, whereas the second class maintains monotonicity, aligning better with physical expectations.

These findings provide insights into the behaviour of stress-strain relations in Neo-Hooke materials and contribute to the understanding of material response under various loading conditions.

Having shown the possible discontinuous nature of multiple solutions with and without symmetry for Rivlin's cube problem, it however remains open whether this can be observed in an experimental setup. It is then natural to inquire as to whether the choice of another elastic energy does not exhibit this surprising response. In other words, this would mean that such insufficiencies stem from the restrictions on the class of elastic energies.

Appendix A. General notation

Inner product For $a,b \in \mathbb{R}^n$ we let $\langle a,b\rangle_{\mathbb{R}^n}$ denote the scalar product on $\mathbb{R}^n$ with associated vector norm $\|a\|^2_{\mathbb{R}^n} = \langle a,a\rangle_{\mathbb{R}^n}$. We denote by $\mathbb{R}^{n\times n}$ the set of real $n\times n$ second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{R}^{n\times n}$ is given by $\langle X,Y\rangle_{\mathbb{R}^{n\times n}} = \mathrm{tr}(XY^\top)$, where the superscript $^\top$ is used to denote transposition. Thus, the Frobenius tensor norm is $\|X\|^2 = \langle X,X\rangle_{\mathbb{R}^{n\times n}}$, where we usually omit the subscript $\mathbb{R}^{n\times n}$ in writing the Frobenius tensor norm. The identity tensor on $\mathbb{R}^{n\times n}$ will be denoted by $\mathbb{1}$, so that $\mathrm{tr}(X) = \langle X,\mathbb{1}\rangle$.

Frequently used spaces

- $\mathrm{Sym}(n)$,$\mathrm{Sym}^+(\mathrm{n})$ and $\mathrm{Sym}^{++}(n)$ denote the symmetric, positive semi-definite symmetric and positive definite symmetric tensors respectively.
- $\mathrm{GL}(n) := \{X \in \mathbb{R}^{n\times n} \,|\, \det X \neq 0\}$ denotes the general linear group.
- $\mathrm{GL}^+(n) := \{X \in \mathbb{R}^{n\times n} \,|\, \det X > 0\}$ is the group of invertible matrices with positive determinant.
- $\mathrm{SL}(n) := \{X \in \mathrm{GL}(n) \,|\, \det X = 1\}$.

- $\mathrm{O}(n):=\{X\in\mathrm{GL}(n)\,|\,X^\top X=\mathbb{1}\}$.
- $\mathrm{SO}(n):=\{X\in\mathrm{GL}(n,\mathbb{R})\,|\,X^\top X=\mathbb{1},\det X=1\}$.
- $\mathfrak{so}(3):=\{X\in\mathbb{R}^{3\times3}\,|\,X^\top=-X\}$ is the Lie-algebra of skew symmetric tensors.
- $\mathfrak{sl}(3):=\{X\in\mathbb{R}^{3\times3}\,|\,\mathrm{tr}(X)=0\}$ is the Lie-algebra of traceless tensors.
- The set of positive real numbers is denoted by $\mathbb{R}_+:=(0,\infty)$, while $\overline{\mathbb{R}}_+=\mathbb{R}_+\cup\{\infty\}$.

Frequently used tensors

- $C=F^\top F$ is the right Cauchy-Green strain tensor.
- $B=FF^\top$ is the left Cauchy-Green (or Finger) strain tensor.
- $U=\sqrt{F^\top F}\in\mathrm{Sym}^{++}(3)$ is the right stretch tensor, i.e. the unique element of $\mathrm{Sym}^{++}(3)$ with $U^2=C$.
- $V=\sqrt{FF^\top}\in\mathrm{Sym}^{++}(3)$ is the left stretch tensor, i.e. the unique element of $\mathrm{Sym}^{++}(3)$ with $V^2=B$.
- We also have the polar decomposition $F=RU=VR\in\mathrm{GL}^+(3)$ with an orthogonal matrix $R\in\mathrm{O}(3)$.

Further definitions and conventions

- For $X\in\mathrm{GL}(3)$, $\mathrm{Cof}X=(\det X)X^{-\top}$ is the cofactor of $X\in\mathrm{GL}(3)$, while $\mathrm{Adj}(X)$ denotes the tensor of transposed cofactors.
- For vectors $\xi,\eta\in\mathbb{R}^3$, we have the tensor product $(\xi\otimes\eta)_{ij}=\xi_i\eta_j$.
- For vectors $v=(v_1,v_2,v_3)^\top\in\mathbb{R}^3$, we define
$$\mathrm{diag}\,v=\begin{pmatrix} v_1 & 0 & 0\\ 0 & v_2 & 0\\ 0 & 0 & v_3\end{pmatrix}.$$
- The Fréchet derivative of a function $W:\mathbb{R}^{3\times3}\to\mathbb{R}$ at $F\in\mathbb{R}^{3\times3}$ applied to the tensor-valued increment H is denoted by $\mathrm{D}_F[W(F)].H$. Similarly, $\mathrm{D}_F^2[W(F)].(H_1,H_2)$ is the bilinear form induced by the second Fréchet derivative of the function W at F applied to (H_1,H_2).
- Let $\Omega\subset\mathbb{R}^3$ be a bounded open domain with Lipschitz boundary $\partial\Omega$. The usual Lebesgue spaces of square-integrable functions, vector or tensor fields on Ω with values in $\mathbb{R}$, $\mathbb{R}^3$, $\mathbb{R}^{3\times3}$ or $\mathrm{SO}(3)$, respectively will be denoted by $\mathrm{L}^2(\Omega;\mathbb{R})$, $\mathrm{L}^2(\Omega;\mathbb{R}^3)$, $\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3})$ and $\mathrm{L}^2(\Omega;\mathrm{SO}(3))$, respectively.
- For vector fields $u=(u_1,u_2,u_3)$ with $u_i\in\mathrm{H}^1(\Omega)$, $i=1,2,3$, we define $\mathrm{D}u:=(\mathrm{D}u_1\,|\,\mathrm{D}u_2\,|\,\mathrm{D}u_3)^\top$.

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