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Localization and Multiplicity for Stationary Stokes Systems with Variable Viscosity

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Abstract. In this paper we discuss the localization and the multiplicity of solutions for the stationary Stokes system with variable viscosity and a reaction force term. The results obtained apply to systems with strongly oscillating periodic viscosity and the corresponding homogenized systems.

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Key words: Stokes system, localization and multiplicity of solutions, fixed point method, homogenization.

1 Introduction

In fluid mechanics, Stokes system describes the linear flow of a viscous fluid. The case of a constant viscosity was extensively studied in the literature (see, for instance [10, 14, 27]). The aim of this paper is to study the localization, multiplicity and homogenization for stationary Stokes systems with variable viscosity in bounded domains. The viscosity is a characteristic of the nature of the fluid and in our case it is supposed to depend on the position. This describes the variable properties of the fluid, as for instance in the case of complex multiphase flow. As applications of such flow one can mention a wide range of engineering problems such as power generation, water treatment, water desalination, refrigeration and air conditioning, and carbon capture and sequestration. We may refer the reader

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to [7,11,20] for the Stokes flow with variable viscosity in thin domains, and to [1,2] for homogenization results.

We start our study with the Stokes system (2.1) in pseudo-stress-velocity formulation (see, for instance [5, 6, 16]), modeling the steady-state motion under an external force f, of an incompressible fluid with variable viscosity μ, where u represents the velocity of the fluid and p is the pressure. Our first objective is to study the localization and the multiplicity of the solution for a system similar to (2.1), if the right-hand side in the first equation depends in a nonlinear manner on the velocity (see (2.11)). From the physical point of view, this corresponds to a reaction external force. A first practical application of our study is to find a velocity dependent reaction force which guarantees that each component of the velocity stays bounded between two a priori given bounds. A second application is to find bounds for the velocity when the reaction external force is given.

In general, two essential tools for proving the localization and multiplicity are the maximum principle and Moser-Harnack type inequalities (see [23]). In order to compensate the loss of these tools in the case of Stokes system, the idea is to associate to system (2.1) (and thus to system (2.11)) a diffusion problem. To do so, we notice that in (2.1) the term ∇p can be seen as a correction of the external force f, in order to keep the incompressibility of the fluid. Through the system (2.6) we associate to ∇p the function q, which can be seen as a correction of the velocity u and then we define $w=u+q$, which solves the diffusion problem (2.7) and so w can be interpreted as the recovered velocity for problem (2.1), under the external force f and a constant pressure. Consequently, following this idea, we can match a Stokes system with variable viscosity and a corresponding diffusion problem with diffusion coefficients varying with space (see, for instance [9]). Once we do this, we prove localization and multiplicity results for the diffusion problem, which will give localization and multiplicity results for the recovered velocity of the Stokes system.

Except for the case of the Navier-Stokes system, where the velocity-dependent convective term can be seen as a reaction force, and for the case of the Coriolis force (see, e.g. [8, 17]), up to our knowledge, there are few mathematical studies in the literature on Stokes systems with a more general reaction force. In this paper we first consider nonlinear reaction forces of the type $h(x,u(x)+q(x))$ (see (2.11)). Thus, the external force h is sensitive to the recovered velocity $w=u+q$ incorporating both velocity u and pressure p (the last one by means of q, which is a correction of the velocity). The main result concerning this first system in given in Theorem 3.2. Then, we consider the special case of planar Stokes systems with a constant pressure, namely such that $\nabla p=0$. We start by giving a sufficient condition on the right-hand side, which guarantees that the pressure of