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A Note on the Global Existence in a Fully Parabolic Patlak-Keller-Segel-Navier-Stokes System

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Abstract. This paper is concerned with a fully parabolic Patlak-Keller-Segel-Navier-Stokes system

$$\begin{cases} n_t+\mathbf{u}\cdot\nabla n=\Delta n-\chi\nabla\cdot(n\nabla c), & (x,t)\in\Omega\times(0,\infty),\\ c_t=\Delta c-c+n, & (x,t)\in\Omega\times(0,\infty),\\ \mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla P=\Delta\mathbf{u}+n\nabla c, & (x,t)\in\Omega\times(0,\infty),\\ \nabla\cdot\mathbf{u}=0, & (x,t)\in\Omega\times(0,\infty), \end{cases}$$

where $\Omega\subset\mathbb{R}^2$ is a smoothly bounded domain and the parameter χ is positive. The main aim of this note is to show that if

$$\int_\Omega n_0(x)dx\leq\frac{4\pi}{\chi},$$

then the solution of the above system is global and bounded in time.

AMS subject classifications: 35B35, 35B40, 35B45, 35K55
Key words: Global existence, boundedness, Patlak-Keller-Segel-Navier-Stokes.

1 Introduction

In this paper, we study the following fully parabolic coupled Patlak-Keller-Segel-Navier-Stokes equation modeling chemotaxis in a moving fluid:

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$$\begin{cases} n_t+\mathbf{u}\cdot\nabla n=\Delta n-\chi\nabla\cdot(n\nabla c), & (x,t)\in\Omega\times(0,\infty), \quad (1.1a)\\ c_t=\Delta c-c+n, & (x,t)\in\Omega\times(0,\infty), \quad (1.1b)\\ \mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla P=\Delta\mathbf{u}+n\nabla c, & (x,t)\in\Omega\times(0,\infty), \quad (1.1c)\\ \nabla\cdot\mathbf{u}=0, & (x,t)\in\Omega\times(0,\infty), \quad (1.1d)\\ \dfrac{\partial n}{\partial\nu}=\dfrac{\partial c}{\partial\nu}=0, \quad \mathbf{u}=\mathbf{0}, & (x,t)\in\partial\Omega\times(0,\infty), \quad (1.1e)\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad \mathbf{u}(x,0)=\mathbf{u}_0(x), & x\in\Omega, \quad (1.1f) \end{cases}$$

where $\chi>0,\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and $\partial/\partial\nu$ denotes the derivative with respect to the outward normal vector ν of $\partial\Omega$. The initial data (n_0,c_0,u_0) satisfies the following conditions:

$$\begin{cases} n_0\in L^\infty(\Omega), & n_0(x)\geq 0 \quad \text{in } \Omega,\\ c_0\in W^{1,\infty}(\Omega), & c_0(x)\geq 0 \quad \text{in } \Omega,\\ \mathbf{u}_0\in D(A^\theta), & \theta\in\left(\dfrac{1}{2},1\right), \end{cases} \tag{1.2}$$

where $A=-\mathcal{P}\Delta$ represents the Stokes operator in

$$L^2_\sigma(\Omega):=\{\varphi\in L^2(\Omega;\mathbb{R}^2)\,|\,\nabla\cdot\varphi=0\}$$

with its domain given by

$$D(A):=W^{2,2}(\Omega;\mathbb{R}^2)\cap W^{1,2}_0(\Omega;\mathbb{R}^2)\cap L^2_\sigma(\Omega),$$

and with $\mathcal{P}$ denoting the Helmholtz projection from $L^2(\Omega;\mathbb{R}^2)$ into $L^2_\sigma(\Omega)$ in [14]. In this model, the unknown functions $n=n(x,t)$ and $c=c(x,t)$ respectively denote the density of the cells and the concentration of the chemicals, and the divergence-free vector field $\mathbf{u}=\mathbf{u}(x,t)$ represents the ambient fluid velocity. The first equation describes the time evolution of the cell density subject to chemotaxis-induced aggregation, diffusion caused by random Brownian motion, and transportation by ambient fluid flow $\mathbf{u}$. Since the cells secrete the chemo-attractants, there exists a deterministic relation between the two distributions n and c, which specify this connection through the second equation when the effect of fluid is neglected. The third equation on the divergence-free vector field $\mathbf{u}$ describes the fluid motion subject to forcing induced by the cells. The reasoning behind the coupling $n\nabla c$ is that in order to make the cells move without acceleration, the fluid exerts frictional force on the moving cells, so reaction forces act on the fluid. The force $n\nabla c$ in the Navier-Stokes equation matches the aggregation nonlinearity in the cell density evolution, which occurs in the Nernst-Planck-Navier-Stokes system (see [3]).