Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2025-0003

Vol. **4**, No. 2, pp. 202-233
May 2025

Local Well-Posedness and Weak-Strong Uniqueness to the Incompressible Vlasov-MHD System

Fucai Li and Jinkai Ni*

School of Mathematics, Nanjing University, Nanjing 210093, P. R. China.

Received 9 April 2025; Accepted 24 April 2025

Abstract. In this paper, we investigate the local well-posedness of strong solutions and weak-strong uniqueness property to the incompressible Vlasov-magnetohydrodynamic (Vlasov-MHD) model in $\mathbb{R}^3_x$. This model consists of a Vlasov equation and the incompressible MHD equations, which interact via the so-called Lorentz force. We first establish the local well-posedness of a strong solution (f,u,B) by utilizing the delicate energy method for the iteration sequence of approximate solutions, provided that the initial data (f_0,u_0,B_0) are H^2-regular and $f_0(x,v)$ has a compact support in the velocity v. We further demonstrate the weak-strong uniqueness property of solutions if $f_0(x,v)\in L^1\cap L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)$, and thereby establish a rigorous connection between the strong and weak solutions to the Vlasov-MHD system. The absence of a dissipation structure in the Vlasov equation and the presence of the strong trilinear coupling term $((u-v)\times B)f$ in the model pose significant challenges in deriving our results. To address these issues, we employ the method of characteristics to estimate the size of the support of f, which enables us to overcome the difficulties associated with evaluating the integral $\int_{\mathbb{R}^3}((u-v)\times B)f\mathrm{d}v$.

AMS subject classifications: 35Q83, 76W05, 35B35

Key words: Incompressible Vlasov-MHD system, local well-posedness, weak-strong uniqueness property, energy method.

---

*Corresponding author. *Email addresses:* `602023210006@smail.nju.edu.cn` (J.-K. Ni), `fli@nju.edu.cn` (F.-C. Li)

1 Introduction and main results

1.1 Previous literature on our model

Kinetic-magnetohydrodynamic (MHD) models have been developed over decades to describe the interactions between the bulk MHD fluids and collisionless energetic particles in magnetized plasmas. These models reveal the evolution of particle motion coupled with fluid dynamics and elucidate the interaction between microscopic particles and macroscopic fluid phenomena. Now kinetic-MHD theories have been widely applied in various fields, including plasma physics [5], engineering [34], biomedicine [1], combustion [33], medical treatment [28], and others. It should be pointed out that although linear kinetic-MHD models align well with physical observations through simulations and analytical studies [7], their nonlinear versions exhibit critical limitations, such as the lack of energy conservation [32]. Thus, it is necessary to develop new kinetic-MHD models to overcome these limitations.

Recently, Cheng *et al.* [6] presented a Vlasov-MHD model

$$\begin{cases} \partial_t f + v\cdot\nabla f + \big((v-u)\times B\big)\cdot\nabla_v f = 0, \\ \partial_t u + u\cdot\nabla u + \nabla P - (\nabla\times B)\times B - \mu_1\Delta u = \int_{\mathbb{R}^3}\big((u-v)\times B\big) f\,\mathrm{d}v, \\ \partial_t B - \nabla\times(u\times B) - \mu_2\Delta B = 0, \\ \nabla\cdot u = 0, \quad \nabla\cdot B = 0 \end{cases} \tag{1.1}$$

with the initial data

$$f(0,x,v) = f_0(x,v), \quad u(0,x) = u_0(x), \quad B(0,x) = B_0(x). \tag{1.2}$$

Here, $f(t,x,v)$ is the particle distribution function at time $t\geq 0, u(t,x)$ is the fluid velocity field, and $B(t,x)$ represents the magnetic field. Besides, $P(t,x)$ represents the pressure of the fluid. The constant $\mu_1>0$ denotes the fluid viscosity coefficient, while $\mu_2>0$ signifies the magnetic diffusion coefficient. The distribution function f is coupled with the magnetic field B and the velocity field u through the nonlinear Lorentz force $(v-u)\times B$.

The system (1.1) obeys the conservations of density, total momentum and total energy (by ignoring the dissipation terms). Cheng *et al.* [6] described how to formally derive this system from the so-called Vlasov-Maxwell system for two-species particles and presented the global existence of weak solutions to (1.1) in the periodic domain $\mathbb{T}^3$ when the initial data $u_0\in L^2(\mathbb{T}^3), B_0\in L^2(\mathbb{T}^3)$ and $f_0\in L^1\cap L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)$, in the same spirit of [30].