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Suppression of Chemotactic Singularity via Poiseuille Flow in a Self-Consistent Patlak-Keller-Segel-Navier-Stokes System

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Abstract. In this paper, we investigate a fully parabolic Patlak-Keller-Segel-Navier-Stokes system with a self-consistent mechanism near the Poiseuille flow $A(y^2,0)$ in $\mathbb{T}\times\mathbb{R}$, which is more natural than the Couette flow from a biomathematical perspective. We demonstrate that the solution to this system maintains global regularity, provided the amplitude A is suitably large and the non-zero modes of the initial chemical density and vorticity are suitably small. To avoid the complex study of the spectral properties of the linear operator and its resolvent, we prove our result using a straightforward weighted energy method combined with a bootstrap argument.

AMS subject classifications: 35K55, 35Q92, 35Q35, 92C17

Key words: Suppression of chemotactic singularity, Poiseuille flow, self-consistent, fully parabolic Patlak-Keller-Segel-Navier-Stokes system.

1 Introduction

In this paper, we delve into the dynamics of the fully parabolic-parabolic self-consistent Patlak-Keller-Segel-Navier-Stokes (PKS-NS) system represented by the following equations:

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$$\begin{cases} \partial_t n - \Delta n + \mathbf{v}\cdot\nabla n = -\nabla\cdot(n\nabla c) + \nabla\cdot(n\nabla\phi), & \text{(1.1a)} \\ \partial_t c - \Delta c + \mathbf{v}\cdot\nabla c = n - c, & \text{(1.1b)} \\ \partial_t \mathbf{v} - \Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + n\nabla c - n\nabla\phi, & \text{(1.1c)} \\ \nabla\cdot\mathbf{v} = 0 & \text{(1.1d)} \end{cases}$$

in the boundary-less domain $\mathbb{T}\times\mathbb{R}$ with $\mathbb{T}=[0,2\pi)$ representing a periodic interval. The system is subject to initial conditions

$$(n,c,\mathbf{v})|_{t=0} = (n_{in}, c_{in}, \mathbf{v}_{in}).$$

The PKS-NS system (1.1) models the intricate interplay between chemotactic movement and a surrounding fluid medium. Here, n denotes the density of bacteria, c represents the chemical concentration, $\mathbf{v}$ and p signify the fluid velocity field and the corresponding pressure, respectively, while ϕ denotes a predetermined potential function arising from gravitational effects.

When $\mathbf{v}=\nabla\phi=\mathbf{0}$, the Eqs. (1.1a) and (1.1b) reduce to the classical parabolic-parabolic Patlak-Keller-Segel (PKS) system

$$\begin{cases} \partial_t n - \Delta n = -\nabla\cdot(n\nabla c), & \text{(1.2a)} \\ \partial_t c - \Delta c = n - c. & \text{(1.2b)} \end{cases}$$

This model, initially proposed by Patlak [51] and further developed by Keller-Segel [34], captures essential dynamics in populations of bacteria. Eq. (1.2a) describes bacterial movement guided by the steepest increase in chemical stimulus concentration, akin to a Brownian motion influenced by external factors. Eq. (1.2b) incorporates the secretion of chemicals by cells themselves, accounting for diffusion and decay. Although system (1.2) possesses straightforward formulations, it exhibits intricate dynamics, garnering significant research interest over the years (see, for instance, [4,6,22,30,44–49,52,58] and references therein). Notably, in two-dimensional settings, Herrero-Velázquez [30] constructed a special blowing-up radial solution with $\|n(\cdot,0)\|_{L^1(\Omega)} > 8\pi$ within a disk $\Omega\subset\mathbb{R}^2$ under no-flux boundary conditions. Similar solutions can be extended to the entire plane $\mathbb{R}^2$, as discussed by [44]. However, the presence of a finite initial mass of bacterial population suppresses finite-time chemotactic singularity: Any solution with $\|n(\cdot,0)\|_{L^1(\mathbb{R}^2)} \le 8\pi$ persists globally in time [4,44]. It is worth noting that addressing blowup phenomena proves substantially more challenging for the fully parabolic-parabolic PKS system (1.2) than its parabolic-elliptic simplification

$$\begin{cases} \partial_t n - \Delta n = -\nabla\cdot(n\nabla c), \\ 0 - \Delta c = n - c \end{cases} \tag{1.3}$$