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Compact Embeddings and Pitt's Property for Weighted Sequence Spaces of Sobolev Type

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Abstract. In this article we introduce a new class of weighted sequence spaces of Sobolev type and prove several compact embedding theorems for them. It is our contention that the chosen class is general enough so as to allow applications in various areas of mathematics and mathematical physics. In particular, our results constitute a generalization of those compact embeddings recently obtained in relation to the spectral analysis of a class of master equations arising in non-equilibrium statistical mechanics. As a byproduct of our considerations, we also prove a theorem of Pitt's type asserting that under some conditions every linear bounded operator acting between such weighted spaces is compact.

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Key words: Sobolev sequence spaces, Pitt's property.

1 Introduction

The essential role played by compact embeddings of Sobolev spaces of various kinds in the analysis of initial- and boundary-value problems for ordinary and partial differential equations is well known (see, e.g. [1,7] and the numerous references therein). Of equal importance are certain Hilbert spaces of sequences and their relation to Sobolev spaces of periodic functions as in [3]. In Section 2

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of this article we introduce a new scale of weighted sequence spaces of Sobolev type and prove compact embedding results for them. The chosen class is general enough so as to allow applications in various areas of mathematics or mathematical physics. In particular, our results generalize those embedding properties recently used in [13] in relation to the analysis of a class of master equations with non-constant coefficients arising in non-equilibrium statistical mechanics, thereby extending the investigations started in [4,5]. We also prove there a theorem of Pitt's type asserting that under some restrictions, every linear bounded operator acting between such weighted spaces is compact. We refer the reader to [11] for the original statement involving a linear bounded operator between two spaces of summable sequences, to [6,8] for much shorter proofs thereof and to [2, Theorem 2.1.4] or [10, Proposition 2.c.3] for yet more condensed arguments.

2 The results

With $s\in[1,+\infty),k\in\mathbb{R}$ and $w=(w_{\mathsf{m}})_{\mathsf{m}\in\mathbb{Z}}$ a sequence of weights satisfying $w_{\mathsf{m}}>0$ for every m, let us consider the separable Banach space $h^{k,s}_{\mathbb{C},w}$ of Sobolev type consisting of all complex sequences $\mathsf{p}=(p_{\mathsf{m}})$ endowed with the usual algebraic operations and the norm

$$\|\mathsf{p}\|_{k,s,w}:=\left(\sum_{\mathsf{m}\in\mathbb{Z}} w_{\mathsf{m}}(1+|\mathsf{m}|^s)^k|p_{\mathsf{m}}|^s\right)^{\frac{1}{s}}<+\infty. \tag{2.1}$$

If $k=0$ we simply write $l^s_{\mathbb{C},w}:=h^{0,s}_{\mathbb{C},w}$ and

$$\|\mathsf{p}\|_{s,w}:=\left(\sum_{\mathsf{m}\in\mathbb{Z}} w_{\mathsf{m}}|p_{\mathsf{m}}|^s\right)^{\frac{1}{s}} \tag{2.2}$$

for the corresponding norm. We may refer to s as the degree of summability of p and to k as its generalized order of differentiability, a terminology justified by the analogy with the usual Sobolev space theory and its relation to Fourier analysis on $\mathbb{R}^d$ (see, e.g. [14, Chapter VI]). Unless $s=1$, the spaces $h^{k,s}_{\mathbb{C},w}$ are reflexive and it is also easily determined that $h^{k,s}_{\mathbb{C},w}$ is a Hilbert space if and only if $s=2$, in which case (2.1) is related to the sesquilinear form

$$(\mathsf{p},\mathsf{q})_{k,2,w}:=\sum_{\mathsf{m}\in\mathbb{Z}} w_{\mathsf{m}}(1+\mathsf{m}^2)^k p_{\mathsf{m}}\bar{q}_{\mathsf{m}}$$