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Stochastic Chebyshev-Picard Iteration Method for Nonlinear Differential Equations with Random Inputs

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Abstract. This work presents a stochastic Chebyshev-Picard iteration method to efficiently solve nonlinear differential equations with random inputs. If the nonlinear problem involves uncertainty, we need to characterize the uncertainty by using a few random variables. The nonlinear stochastic problems require solving the nonlinear system for a large number of samples in the stochastic space to quantify the statistics of the system of response and explore the uncertainty quantification. The computational cost is very expensive. To overcome the difficulty, a low rank approximation is introduced to the solution of the corresponding nonlinear problem and admits a variable-separation form in terms of stochastic basis functions and deterministic basis functions. No iteration is performed at each enrichment step. These basis functions are model-oriented and involve offline computation. To efficiently identify the stochastic basis functions, we utilize the greedy algorithm to select some optimal samples. Then the modified Chebyshev-Picard iteration method is used to solve the nonlinear system at the selected optimal samples, the solutions of which are used to train the deterministic basis functions. With the deterministic basis functions, we can obtain the corresponding stochastic basis functions by solving linear differential systems. The computation of the stochastic Chebyshev-Picard method decomposes into an offline phase and an online phase. This is

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very desirable for scientific computation. Several examples are presented to illustrate the efficacy of the proposed method for different nonlinear differential equations.

AMS subject classifications: 65C20, 65L60, 65M70

Key words: Nonlinear ordinary differential equation, Chebyshev-Picard iteration method, variable-separation method, random inputs.

1 Introduction

In recent years, nonlinear dynamical systems arise in many fields of science, particular in physics, chemistry, finance and engineering problems. The real-world models usually contain some uncertainties because of lacking enough knowledge about physical properties and measurement noise. Therefore, such complex models with uncertainties can be described by the nonlinear ordinary differential equations with random inputs (e.g., model coefficients, external loadings or the initial conditions). For deterministic nonlinear systems, there exist a broad class of approaches of analytical, semi-analytical and numerical methods.

The most commonly used methods are Adomian decomposition method (ADM) [2, 19, 41], the variational iteration method (VIM) [20, 21], the homotopy perturbation method (HPM) [1], the homotopy analysis method (HAM) [9] and also other numerical methods such as one shot method [7], Runge-Kutta-Nyström method [3, 25], Hilber-Hughes-Taylor-α method [22] and multistep, or predictor-corrector methods [23]. On the other hand, collocation methods using Chebyshev and Legendre as Jacobi polynomials are presented in [4, 11, 15, 33]. To solve for the evolution of a nonlinear dynamical system beyond a certain time instant t_0, the most straightforward approaches could be the direct difference method, such as Runge-Kutta method. For nonlinear deterministic differential equations, there are mathematical theories and computational methods. The consideration of uncertainty in modelling has experienced a significant increase over the last few years. For different random problems, some new methods, such as bi-orthogonal PINN method, stochastic homogenization method and novel partitioned time-stepping method, are proposed in [31, 46, 48].

By estimating the statistical properties with Monte Carlo simulation, some numerical schemes have been obtained which have been called Runge-Kutta Monte Carlo methods. Quasi-Monte Carlo (QMC) methods [10, 14] are deterministic version of Monte Carlo (MC) methods, in the sense that the random samples