Commun. Math. Res.
doi: 10.4208/cmr.2024-0017

Vol. **40**, No. 3, pp. 313-342
August 2024

Convergence Analysis of Kernel Learning FBSDE Filter

Yunzheng Lyu* and Feng Bao

Department of Mathematics, Florida State University, FL 32306, USA.

Received 22 May 2024; Accepted 14 August 2024

Abstract. Kernel learning forward backward stochastic differential equations (FBSDE) filter is an iterative and adaptive meshfree approach to solve the nonlinear filtering problem. It builds from forward backward SDE for Fokker-Planker equation, which defines evolving density for the state variable, and employs kernel density estimation (KDE) to approximate density. This algorithm has shown more superior performance than mainstream particle filter method, in both convergence speed and efficiency of solving high dimension problems. However, this method has only been shown to converge empirically. In this paper, we present a rigorous analysis to demonstrate its local and global convergence, and provide theoretical support for its empirical results.

AMS subject classifications: 93E11, 65B99, 65C05, 65C35, 65C60

Key words: Forward backward stochastic differential equations, kernel density estimation, nonlinear filtering problems, convergence analysis.

1 Introduction

Optimal filtering problem is to estimate unknown underlying state variables S_t from associated noisy observation data O_t. In the general application setting, prior distribution is known for the underlying state system, combined with likelihood function relating state variables to observation data, one can perform Bayesian inference to estimate evolving posterior density for the unknown state. It has applications in many fields, ranging from signal processing, quantum physics, mathematical finance to machine learning and AI etc.

*Corresponding author. *Email addresses:* `yl19@fsu.edu` (Y. Lyu), `fbao@fsu.edu` (F. Bao)

When both the state and observation systems are linear, Kalman filter provides optimal analytical solution for posterior state density [16]. To cope with nonlinear filtering problems, different variations of Kalman filter have been proposed. Extended Kalman filter linearly approximates nonlinear transformations by applying tangent linear operator or Jacobian matrix [14]. Ensemble Kalman filter generates ensemble of samples to approximate covariance for nonlinear system [11]. These variations within the Kalman filter framework perform badly when non-linearity is high.

Following the work of bootstrap filter [12], particle filter [10] or sequential Monte Carlo [17] simulate posterior state density by propagating samples through a sequence of importance sampling, resampling and Markov Chain Monte Carlo (MCMC) steps. Various improvements have been made later to refine these intermediate sub-steps, e.g. auxiliary particle filter [19] and population Monte Carlo [13]. This framework can handle nonlinear system effectively, but it suffers degeneracy issue in long term or high frequency simulations.

Another line of work calculates posterior state density analytically through SPDE [15,21]. Their major drawback is slow convergence and profound complexity, especially for high-dimensional problems. Due to the equivalence found between Forward Backward Doubly Stochastic Differential Equation (FBDSDE) and certain parabolic SPDE [18], FBDSDE framework is established to estimate posterior state density [2–8]. Archibald and Bao [1] further simplifies this framework to FBSDE filter algorithm, in which only prior state density is derived through the FBSDE system, sample posterior density is then approximated through Bayesian inference and posterior density function is learnt from sample results by kernel learning methods. This framework is mesh free and can deal with high-dimensional problems efficiently.

Along the development of algorithms, convergence analysis of filtering algorithms is more subtle. Crisan and Doucet [9] prove the convergence of a general particle filter under suitable regularity conditions. In this paper, we follow similar convergence analysis framework and show the convergence results for FBSDE filter.

The rest of paper is organized as follows. In Section 2, we summarize the theoretical background for FBSDE filter. In Section 3, we demonstrate implementations of FBSDE filter algorithm. In Section 4, we conduct convergence analysis for the algorithm. And concluding remarks are given in Section 5.

2 FBSDE filter

Underlying problems in FBSDE filters take stochastic system form