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The General Dabrowski-Sitarz-Zalecki Type Theorem for Odd Dimensional Manifolds with Boundary II

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Abstract. In [T. Wu *et al.*, arXiv2310.09775, 2023], a general Dabrowski-Sitarz-Zalecki type theorem for odd dimensional manifolds with boundary was proved. In this paper, we give the proof of the another general Dabrowski-Sitarz-Zalecki type theorem for the spectral Einstein functional associated with the Dirac operator on odd dimensional manifolds with boundary.

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Key words: Dirac operators, spectral Einstein functional, Dabrowski-Sitarz-Zalecki type theorems.

1 Introduction

The noncommutative residue found in [7, 18] plays a prominent role in noncommutative geometry. For one-dimensional manifolds, the noncommutative residue was discovered by Adler [2] in connection with geometric aspects of nonlinear partial differential equations. For arbitrary closed compact n-dimensional manifolds, the noncommutative residue was introduced by Wodzicki [18] using the theory of zeta functions of elliptic pseudo-differential operators. Connes [3] used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Connes [4] showed us that the noncommutative residue on a com-

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pact manifold M coincided with the Dixmier's trace on pseudo-differential operators of order $-\dim M$. And Connes claimed that the noncommutative residue of the square of the inverse of the Dirac operator was proportioned to the Einstein-Hilbert action. Kastler [9] gave a brute-force proof of this theorem. Kalau and Walze [8] proved this theorem in the normal coordinates system simultaneously. Ackermann [1] proved that the Wodzicki residue of the square of the inverse of the Dirac operator $\mathrm{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of D^2.

On the other hand, Wang [13, 14] generalized the Connes' results to the case of manifolds with boundary, and proved the Kastler-Kalau-Walze type theorem for the Dirac operator and the signature operator on lower-dimensional manifolds with boundary [15]. Wang [15, 16] computed $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-1}]$ and $\widetilde{\mathrm{Wres}}[\pi^+D^{-2}\circ\pi^+D^{-2}]$, where the two operators are symmetric, in these cases the boundary term vanished. But for $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-3}]$, the authors got a non-vanishing boundary term [11], and gave a theoretical explanation for gravitational action on boundary. The authors defined bilinear functionals of vector fields and differential forms, the densities of which yielded the metric and Einstein tensors on even dimensional Riemannian manifolds [5]. In [20], the authors computed the generalized noncommutative residue $\widetilde{\mathrm{Wres}}[\pi^+(c(X)D^{-1})\circ\pi^+(D^{-(2m-2)})]$, $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}D^{-1})\circ\pi^+(D^{-(2m-1)})]$, $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}D^{-2})\circ\pi^+(D^{-(2m-2)})]$ on odd dimensional manifolds with boundary. Wu and Wang [19] defined the spectral Einstein functional associated with the Dirac operator for manifolds with boundary, and computed the noncommutative residue $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})\circ\pi^+(D^{-(n-1)})]$ and $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-2})\circ\pi^+(D^{-(n-2)})]$ on n-dimensional compact manifolds, n is even. They also computed $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-2})\circ\pi^+(D^{-(n-1)})]$ for n-dimensional manifolds with boundary, n is odd. That is, they computed $\widetilde{\mathrm{Wres}}[\pi^+P_1\circ\pi^+P_2]$ for n-dimensional manifolds, n is odd and $n+order(P_1)+order(P_2)=1$. The motivation of this paper is to prove the another general Dabrowski-Sitarz-Zalecki type theorem associated with the Dirac operator for odd dimensional manifolds with boundary. So we want to compute $\mathrm{Wres}[\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-n}]$ when $\dim M=n$ and n is odd as in [5]. But $\mathrm{Wres}[\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-n}]=0$. When M is a manifold with the even dimensional boundary, we hope to compute $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})\circ\pi^+(D^{-(n-1)})]$ and $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-2})\circ\pi^+(D^{-(n-2)})]$ as in [19] and get the non-zero boundary term, but through the computations, the boundary term is still zero. The reason is that $\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-n}$ maps $S^{\pm}(TM)$ to $S^{\mp}(TM)$