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A High-Order Meshless Energy-Preserving Algorithm for the Beam Equation

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Abstract. In this paper, a meshless energy-preserving algorithm which can be arbitrarily high-order in temporal direction for the beam equation has been proposed. Based on the method of lines, we first use the radial basis function quasi-interpolation method to discretize spatial variable and obtain a semi-discrete Hamiltonian system by using the premultiplication of a diagonal matrix. Then, symplectic Runge-Kutta method that can conserve quadratic invariants exactly has been used to discretize the temporal variable, which yields a fully discrete meshless scheme. Due to the specific quadratic energy expression of the beam equation, the proposed meshless scheme here is not only energy-preserving but also arbitrarily high-order in temporal direction. Besides uniform and nonuniform grids, numerical experiments on random grids are also conducted, which demonstrate the properties of the proposed scheme very well.

AMS subject classifications: 37K05, 65L06, 65L12, 65N40

Key words: Beam equation, meshless scheme, energy-preserving method, radial basis function, symplectic Runge-Kutta method.

1 Introduction

As one of the most commonly used components which makes up various structures in engineering, beams have a very long history. It can be traced back to Leonardo da Vinci (1452-1519) and Galileo (1584-1642), which was popularized

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and developed in the 18th century by Euler, Jacobi and Bernoulli. In order to avoid accidents caused by resonance, it is necessary to study the transverse vibration of such structures to achieve structural safety and long-term stability in engineering design. With the development of new technologies and materials, beams have been widely used in civil engineering, bridges, ships, aerospace and other communities because of their high strength and low mass [15,23,47].

When a beam undergoes transverse vibration, due to the appearance of shear stress and torsion moment, the equation appears as a fourth derivative term and the boundary conditions also increase. The general form of the beam equation is as follows:

$$u_{tt}+\Delta^2 u=f(u), \tag{1.1}$$

where Δ is the Laplace operator and $f(u)=F'(u)$. Here, $f(u)$ is a linear real function which guarantees that the energy expression of system (1.1) is quadratic.

The beam equation had been studied by many scholars in recent decades. Gupta [17,18], Li and Zhang [32] analyzed and studied the existence and uniqueness of the solution for the beam equation in literatures. Qin [48], Ma and Martinez [38] presented a study on the positive solution of the fourth-order beam equation. Thankane and Styš [46] studied the beam equation with the free end by using the finite difference method. Gunakala *et al.* [16] studied the beam equation with the finite element method. Li and Yang [31] constructed and analyzed a compact difference scheme for the two-dimensional beam equation. For more information about the beam equation, please refer to [8, 22, 37] and references therein.

Numerical algorithms are called structure-preserving algorithms if they can preserve some internal properties of the original system [9]. In recent years, alternatives of the symplectic integrators, including symplectic neural networks, have become increasingly popular [3, 27]. The structure-preserving algorithm [10, 14, 35, 55] cannot only preserve the geometric structure but also preserve algebraic properties and physical conservation laws. As a noticeable structure of the Hamiltonian system, energy has attracted the attention of many scholars [4,19,49].

In the past decades, many methods have been proposed to solve the numerical solutions for partial differential equations (PDEs) to get the energy-preserving algorithms, such as the finite element method [25], the finite difference method [24, 50], the Fourier pseudo-spectral method [5, 6], the discontinuous Galerkin method [36,45], the Gauss-Legendre collocation method [34] and some methods which can conserve more general energy, for example, the Lagrange multiplier approach [33], the implicit-explicit relaxation Runge-Kutta method [7], the relaxation exponential rosenbrock-type method [29] and so on. However, all of