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Eigenvalue Problem for a Class of Quasilinear Elliptic Operators with Mixed Boundary Value Condition in a Variable Exponent Sobolev Space

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Abstract. In this paper, we consider an eigenvalue problem for a class of nonlinear elliptic operators containing $p(\cdot)$-Laplacian and mean curvature operator with mixed boundary conditions. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on another part of the boundary. Using the Ljusternik-Schnirelmann variational method, we show the existence of infinitely many positive eigenvalues of the equation. Furthermore, under some conditions, we derive that the infimum of the set of all the eigenvalues becomes zero or remains to be positive.

AMS subject classifications: 49R50, 35A01, 35J62, 35J57

Key words: Eigenvalue problem, $p(\cdot)$-Laplacian type equation, mean curvature operator, mixed boundary value problem.

1 Introduction

In this paper, we consider the following eigenvalue problem with mixed boundary conditions:

$$\begin{cases} -\operatorname{div}\left[\boldsymbol{a}(x,\nabla u(x))\right]=f\big(x,u(x)\big) & \text{in } \Omega, \\ u(x)=0 & \text{on } \Gamma_1, \\ \boldsymbol{n}(x)\cdot\boldsymbol{a}\big(x,\nabla u(x)\big)=\lambda g\big(x,u(x)\big) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

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Here Ω is a bounded domain of $\mathbb{R}^N$ $(N\geq 2)$ with a Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are non-empty disjoint open subsets of } \Gamma \text{ such that } \overline{\Gamma_1}\cup\overline{\Gamma_2}=\Gamma, \tag{1.2}$$

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. Here we say that $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$, if for a.e. $x\in\Omega$, the map $\mathbb{R}^N\ni\boldsymbol{\xi}\mapsto\boldsymbol{a}(x,\boldsymbol{\xi})$ is continuous and for every $\boldsymbol{\xi}\in\mathbb{R}^N$, the map $\Omega\ni x\mapsto\boldsymbol{a}(x,\boldsymbol{\xi})$ is measurable on Ω. The operator $u\mapsto\operatorname{div}[\boldsymbol{a}(x,\nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian $\Delta_{p(x)}u(x)=\operatorname{div}[|\nabla u(x)|^{p(x)-2}\nabla u(x)]$ and the mean curvature operator $\operatorname{div}[(1+|\nabla u(x)|^2)^{(p(x)-2)/2}\nabla u(x)]$. This generality brings about difficulties and requires some conditions.

We impose mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g:\Gamma_2\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying some structure conditions and λ is a real number.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [44]), in electrorheological fluids (Diening [14], Halsey [25], Mihăilescu and Rădulescu [32], Růžička [37]).

However, since we find a few papers associated with the problem with the mixed boundary conditions in variable exponent Sobolev space as in (1.1) (for example, Aramaki [2–4, 10]), we are convinced of the reason for existence of this paper.

When $p(x)\equiv p=\text{const.}$, there are many articles for the p-Laplacian. For example, see Lê [28], Anane [1], Friedlander [22]. For the p-Laplacian Dirichlet eigenvalue problem

$$\begin{cases} -\Delta_p u(x)=\lambda|u(x)|^{p-2}u(x) & \text{in } \Omega, \\ \ \ u(x)=0 & \text{on } \Gamma \end{cases}$$

it is well-known that the following properties hold:

(1) There exists a nondecreasing sequence of positive eigenvalues $\{\lambda_n\}$ tending to ∞ as $n\to\infty$.

(2) The first eigenvalue λ_1 is simple and only the eigenfunction associated with λ_1 do not change sign.