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The m-CMP Inverse in Minkowski Space

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Abstract. In this paper, a new generalized m-CMP inverse is introduced into the Minkowski space, and its related properties are discussed. Secondly, three different expressions of m-CMP inverse are listed, namely Hartwig-Spindelböck decomposition, full-rank decomposition and integral expression. Finally, the application of the inverse m-CMP equation in solving linear equations is given.

AMS subject classifications: 15A10

Key words: Minkowski space, m-CMP inverse, Hartwig-Spindelböck decomposition, full-rank factorization, integral expression, system of linear equations.

1 Introduction

Generalized inverse theory plays an important role in many aspects, such as operator theory, Markov chain, and even can be applied to statistics, cryptography, etc. Moore-Penrose inverse and Drazin inverse are the two most classical generalized inverses, which have been deeply studied by many scholars from different directions and angles. In 1902, Moore described it with five equations, and later Penrose described it with four equations, which later became known as the Moore-Penrose inverse [25]. Let $A\in\mathbb{C}^{m\times n}$. Then the matrix $X\in\mathbb{C}^{n\times m}$ verifying

$$AXA=A,\quad XAX=X,\quad (AX)^*=AX,\quad (XA)^*=XA$$

is called the Moore-Penrose inverse of A, denoted by $A^\dagger$. In addition, if X satisfies $XAX=X$, then we call X an outer inverse of A. For subspaces $\mathcal{T}\subseteq\mathbb{C}^n$ and $\mathcal{S}\subseteq$

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$\mathbb{C}^m$, an outer inverse X of A with $\mathcal{R}(X)=\mathcal{T}$ and $\mathcal{N}(X)=\mathcal{S}$ is unique, and is denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$. In 1958, Drazin [4] proposed pseudo-inverses in semigroups and associative rings. Let $A\in\mathbb{C}^{n\times n}$ with $k=\operatorname{ind}(A)$. Then the matrix $X\in\mathbb{C}^{n\times n}$ satisfying

$$XAX=X, \quad AX=XA, \quad A^{k+1}X=A^k$$

is called the Drazin inverse of A, denoted by A^D. In the case, $\operatorname{ind}(A)=1$, the Drazin inverse of A reduces to the group inverse of A, which is denoted by $A^{\#}$. With the development of generalized inverse theory, more and more scholars focus on the new generalized inverse established by Moore-Penrose inverse and Drazin inverse expressions. In 2014, Malik [16] proposed a new generalized inverse: Let $A\in\mathbb{C}^{n\times n}$ with $k=\operatorname{ind}(A)$, then the matrix $X\in\mathbb{C}^{n\times n}$ satisfying

$$XAX=X, \quad XA=A^DA, \quad A^kX=A^kA^{\dagger}$$

is called the DMP inverse of A, denoted by $A^{D\dagger}$.

Many scholars have conducted more in-depth studies on DMP inversion. Liu [12] proposed an iterative method for calculating DMP inversion, Ferreyra [5] presented DMP inverse general form, Ma *et al.* [15] and Wang *et al.* [30] gave different representations of DMP inverse, respectively. Kyrchei [11] discussed the solvability of quaternion equations by using the correlation properties of DMP inversions and their duals. Romo [27] generalizes it to finite potent endomorphisms on arbitrary vector spaces. Zhu *et al.* [35] extended it to involutory rings. Mosic *et al.* [22] and Yu *et al.* [29] generalize it to Hilbert space operators.

In 1999, Meenakshi [17] established the Minkowski inverse in Minkowski space: Let $A\in\mathbb{C}^{m\times n}$. Then the matrix $X\in\mathbb{C}^{n\times m}$ satisfying

$$AXA=A, \quad XAX=X, \quad (AX)^{\sim}=AX, \quad (XA)^{\sim}=XA$$

is called the Minkowski inverse of A, denoted by A^m. The Minkowski adjoint $A^{\sim}\triangleq GA^*F$, where G and F are Minkowski metric matrices with orders n and m, respectively. Meenakshi [18] describes the properties and applications of Minkowski inverses. For more discussion of Minkowski's inverse, please refer to [1,9,13].

We also note that the m-core inverse [32], m-core-EP inverse [33], and m-WG inverse [34], recently defined in the Minkowski space are extensions of the core inverse [2], core-EP inverse [26], and weak group inverse [31], respectively. In 2023, Gao *et al.* [6] introduced a new generalized inverse in the Minkowski space and discussed its properties, characterization, representation, and applications, called m-DMP inverse. Let $A\in\mathbb{C}^{n\times n}$ with $k=\operatorname{ind}(A)$ and $\operatorname{rank}(A^{\sim}AA^{\sim})=$