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When the Gromov-Hausdorff Distance Between Finite-Dimensional Space and Its Subset Is Finite?

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Abstract. In this paper we prove that the Gromov-Hausdorff distance between $\mathbb{R}^n$ and its subset A is finite if and only if A is an ε-net in $\mathbb{R}^n$ for some $\varepsilon > 0$. For infinite-dimensional Euclidean spaces this is not true. The proof is essentially based on upper estimate of the Euclidean Gromov-Hausdorff distance by means of the Gromov-Hausdorff distance.

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Key words: Metric space, ε-net, Gromov-Hausdorff distance.

1 Introduction

This paper is devoted to investigation of geometry of the classical Gromov-Hausdorff distance [2–4]. Traditionally, the Gromov-Hausdorff distance is used for bounded metric spaces, mainly, for compact ones. In the case of non-bounded metric spaces, this distance is applied to define the pointed Gromov-Hausdorff convergence, and besides, there were a few attempts to define the corresponding distance function in this case, see for instance [5]. Since the Gromov-Hausdorff distance between isometric metric spaces vanishes, it is natural to identify such spaces in this theory. Thus, the main space for investigating the Gromov-Hausdorff distance is the space $\mathcal{M}$ of non-empty compact metric spaces considered

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upto isometry, endowed with the Gromov-Hausdorff distance. Here the distance function is a metric, and this metric is complete, separable, geodesic, etc., see for details [2,4].

In [4], Gromov described some geometric properties of the Gromov-Hausdorff distance on the space $\mathcal{GH}$ of all non-empty metric spaces, not necessarily bounded, considered up to isometry. It is easy to see that $\mathcal{GH}$ is not a set, but a proper class in terms of von Neumann-Bernays-Gödel set theory. Gromov suggested to consider subclasses consisting of all metric spaces on finite distance between each other. We called such subclasses clouds. Gromov announced that the clouds are obviously complete and contractible [4]. He suggested to see that on the example of the cloud containing $\mathbb{R}^n$. The idea is to consider a mapping of $\mathcal{GH}$ into itself that for each $X \in \mathcal{GH}$, multiplies all distances in X by some real $\lambda > 0$. It is easy to see that such mapping takes $\mathbb{R}^n$ to an isometric metric space, i.e., $\mathbb{R}^n$ is a fixed point of this mapping. However, the Gromov-Hausdorff distances from $\mathbb{R}^n$ to all other metrics spaces in its cloud are multiplied by λ as well. It remains to see what happens when $\lambda \to 0+$.

Nevertheless, these "obvious observations" lead to a few questions. To start with, the clouds are proper classes (B. Nesterov, private conversations); contractibility is a topological notion; so, to speak about contractibility of a cloud, we have to define a topology on it. However, it is not possible to introduce a topology on a proper class, because the proper class cannot be an element of any other class by definition, but each topological space is an element of its own topology. Bogaty and Tuzhilin [1] developed a convenient language that allows to avoid the set-theoretic problems. Namely, they introduced an analogue of topology on so-called set-filtered classes (each set belongs to this family, together with the class $\mathcal{GH}$) and defined continuous mappings between such classes. At the same time, the authors of [1] found examples of metric spaces that jump onto infinite Gromov-Hausdorff distance after multiplying their distance functions on some $\lambda > 0$. Thus, the multiplication on such λ does not map the clouds into themselves. This strange behavior of clouds increased the interest to them, see [1] for details.

In the present paper we continue investigation of the geometry of the standard (non-pointed) Gromov-Hausdorff distance between non-bounded metric spaces. It is well-known that each ε-net of a metric space X is on finite Gromov-Hausdorff distance from X. Is the converse statement true as well? Example 3.1 shows that it is not true even for infinite-dimensional Euclidean spaces. In the present paper we prove that the converse statement holds for finite-dimensional Euclidean spaces. A natural question is to understand what happens for other finite-dimensional normed spaces. It turns out that there are a few obstacles to obtain such generalizations.