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When the Gromov-Hausdorff Distance Between Finite-Dimensional Space and Its Subset Is Finite?

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Abstract. In this paper we prove that the Gromov-Hausdorff distance between $\mathbb{R}^n$ and its subset A is finite if and only if A is an ε-net in $\mathbb{R}^n$ for some $\varepsilon > 0$. For infinite-dimensional Euclidean spaces this is not true. The proof is essentially based on upper estimate of the Euclidean Gromov-Hausdorff distance by means of the Gromov-Hausdorff distance.

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1 Introduction

This paper is devoted to investigation of geometry of the classical Gromov-Hausdorff distance [2–4]. Traditionally, the Gromov-Hausdorff distance is used for bounded metric spaces, mainly, for compact ones. In the case of non-bounded metric spaces, this distance is applied to define the pointed Gromov-Hausdorff convergence, and besides, there were a few attempts to define the corresponding distance function in this case, see for instance [5]. Since the Gromov-Hausdorff distance between isometric metric spaces vanishes, it is natural to identify such spaces in this theory. Thus, the main space for investigating the Gromov-Hausdorff distance is the space $\mathcal{M}$ of non-empty compact metric spaces considered

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upto isometry, endowed with the Gromov-Hausdorff distance. Here the distance function is a metric, and this metric is complete, separable, geodesic, etc., see for details [2,4].

In [4], Gromov described some geometric properties of the Gromov-Hausdorff distance on the space $\mathcal{GH}$ of all non-empty metric spaces, not necessarily bounded, considered up to isometry. It is easy to see that $\mathcal{GH}$ is not a set, but a proper class in terms of von Neumann-Bernays-Gödel set theory. Gromov suggested to consider subclasses consisting of all metric spaces on finite distance between each other. We called such subclasses clouds. Gromov announced that the clouds are obviously complete and contractible [4]. He suggested to see that on the example of the cloud containing $\mathbb{R}^n$. The idea is to consider a mapping of $\mathcal{GH}$ into itself that for each $X\in\mathcal{GH}$, multiplies all distances in X by some real $\lambda>0$. It is easy to see that such mapping takes $\mathbb{R}^n$ to an isometric metric space, i.e., $\mathbb{R}^n$ is a fixed point of this mapping. However, the Gromov-Hausdorff distances from $\mathbb{R}^n$ to all other metrics spaces in its cloud are multiplied by λ as well. It remains to see what happens when $\lambda\to 0+$.

Nevertheless, these "obvious observations" lead to a few questions. To start with, the clouds are proper classes (B. Nesterov, private conversations); contractibility is a topological notion; so, to speak about contractibility of a cloud, we have to define a topology on it. However, it is not possible to introduce a topology on a proper class, because the proper class cannot be an element of any other class by definition, but each topological space is an element of its own topology. Bogaty and Tuzhilin [1] developed a convenient language that allows to avoid the set-theoretic problems. Namely, they introduced an analogue of topology on so-called set-filtered classes (each set belongs to this family, together with the class $\mathcal{GH}$) and defined continuous mappings between such classes. At the same time, the authors of [1] found examples of metric spaces that jump onto infinite Gromov-Hausdorff distance after multiplying their distance functions on some $\lambda>0$. Thus, the multiplication on such λ does not map the clouds into themselves. This strange behavior of clouds increased the interest to them, see [1] for details.

In the present paper we continue investigation of the geometry of the standard (non-pointed) Gromov-Hausdorff distance between non-bounded metric spaces. It is well-known that each ε-net of a metric space X is on finite Gromov-Hausdorff distance from X. Is the converse statement true as well? Example 3.1 shows that it is not true even for infinite-dimensional Euclidean spaces. In the present paper we prove that the converse statement holds for finite-dimensional Euclidean spaces. A natural question is to understand what happens for other finite-dimensional normed spaces. It turns out that there are a few obstacles to obtain such generalizations.

Firstly, our approach is not completely straightforward. In particularly, it is based on [6, Theorem 2] by Memoli. This inequality provides an upper estimate of the Euclidean Gromov-Hausdorff distance in terms of the classical Gromov-Hausdorff distance. The effectiveness of this theorem is based on the richness of the isometry group $\mathrm{Iso}(\mathbb{R}^n)$ of $\mathbb{R}^n$, that is not the case for other finite-dimensional normed spaces. Secondly, we might consider some ε-net σ in $\mathbb{R}^n$ and try to approximate it in terms of the Gromov-Hausdorff distance with an ε'-net σ' in a finite-dimensional normed space X. However, according to paper [7], it holds $\mathrm{d}_{GH}(\sigma,\sigma')=\infty$ unless X is isometric to $\mathbb{R}^n$, so it is also impossible to transfer our result to such X using this idea. Summing up, the question whether or not a subset Y of a finite-dimensional normed space X on a finite Gromov-Hausdorff distance from X is an ε-net in X for some $\varepsilon>0$ remains open.

2 Preliminaries

For an arbitrary metric space X, the distance between its points x and y we denote by $|xy|$. Let $B_r(a)=\{x\in X:|ax|\le r\}$ and $S_r(a)=\{x\in X:|ax|=r\}$ be the closed ball and the sphere of radius r centered at the point a, respectively. For an arbitrary subset $A\subset X$, its closure in X is denoted by $\overline{A}$. For non-empty subsets $A\subset X$ and $B\subset X$, we set $\mathrm{d}(A,B)=\inf\{|ab|:a\in A,b\in B\}$.

Definition 2.1. *Let A and B be non-empty subsets of a metric space X. The Hausdorff distance between A and B is the value*

$$\mathrm{d}_H(A,B)=\inf\big\{r>0:A\subset B_r(B),B\subset B_r(A)\big\}.$$

Definition 2.2. *Let X and Y be metric spaces. If X',Y' are subsets of a metric space Z such that X' is isometric to X and Y' is isometric to Y, then we call the triple (X',Y',Z) a metric realization of the pair (X,Y).*

Definition 2.3. *The Gromov-Hausdorff distance $\mathrm{d}_{GH}(X,Y)$ between two metric spaces X,Y is the infimum of positive numbers r such that there exists a metric realization (X',Y',Z) of the pair (X,Y) with $\mathrm{d}_H(X',Y')\le r$.*

Let X and Y be non-empty sets. Recall that any subset $\sigma\subset X\times Y$ is called a relation between X and Y. Denote the set of all non-empty relations between X and Y by $\mathcal{P}_0(X,Y)$. We set

$$\begin{aligned}&\pi_X\colon X\times Y\to X, \quad \pi_X(x,y)=x,\\ &\pi_Y\colon X\times Y\to Y, \quad \pi_Y(x,y)=y.\end{aligned}$$

Definition 2.4. *A relation $R\subset X\times Y$ is called a correspondence if $\pi_X|_R$ and $\pi_Y|_R$ are surjective. In other words, correspondences are multivalued surjective mappings. Denote the set of all correspondences between X and Y by $\mathcal{R}(X,Y)$.*

Definition 2.5. *Suppose $R\in\mathcal{R}(X,Y)$, and $A\subset X$, $B\subset Y$. We use the following standard notation:*

$$R(A)=\{y\in Y\mid \exists x\in A\colon (x,y)\in R\},$$
$$R^{-1}(B)=\{x\in X\mid \exists y\in B\colon (x,y)\in R\}.$$

Definition 2.6. *Let X,Y be arbitrary metric spaces. Then for every $\sigma\in\mathcal{P}_0(X,Y)$, the distortion of σ is defined as*

$$\operatorname{dis}\sigma=\sup\Big\{\big||xx'|-|yy'|\big|:(x,y),(x',y')\in\sigma\Big\}.$$

Claim 1 ([2,8])**.** For arbitrary metric spaces X and Y, the following equality holds:

$$2\mathrm{d}_{GH}(X,Y)=\inf\big\{\operatorname{dis}R:R\in\mathcal{R}(X,Y)\big\}.$$

Recall that for any sets X,Y,Z, and relations $\sigma_1\in\mathcal{P}_0(X,Y)$, $\sigma_2\in\mathcal{P}_0(Y,Z)$, the composition of σ_1 and σ_2, denoted by $\sigma_2\circ\sigma_1$, is the set of all $(x,z)\in X\times Z$ for which there exists $y\in Y$ such that $(x,y)\in\sigma_1$ and $(y,z)\in\sigma_2$.

Claim 2 ([2,8])**.** Let X,Y,Z be metric spaces, $R_1\in\mathcal{R}(X,Y),R_2\in\mathcal{R}(Y,Z)$. Then $R_2\circ R_1\in\mathcal{R}(X,Z)$ and the following inequality holds:

$$\operatorname{dis}(R_2\circ R_1)\le\operatorname{dis}R_1+\operatorname{dis}R_2.$$

Claim 3. Given non-empty subsets A and B of a metric space X with $\mathrm{d}_H(A,B)<c$, the set $U=\big\{(a,b)\in A\times B:|ab|<c\big\}$ is a correspondence between A and B such that $\operatorname{dis}U\le 2c$.

Proof. The inequality $\mathrm{d}_H(A,B)<c$ implies that for an arbitrary $a\in A$, there exists $b\in B$ such that $|ab|<c$. Hence, $(a,b)\in U$ and, therefore, the projection of U to A is surjective. Similarly, the projection of U to B is surjective. Thus, U is a correspondence between A and B.

Given $(a,b),(a',b')\in U$ with $|ab|<c$, $|a'b'|<c$, we get

$$\big||aa'|-|bb'|\big|\le|ab|+|a'b'|=2c.$$

Thus, $\operatorname{dis}U\le 2c$. □

Claim 4 ([2,8])**.** Let X and Y be metric spaces, and the diameter of one of them is finite. Then

$$\mathrm{d}_{GH}(X,Y) \geq \frac{1}{2}|\mathrm{diam}X - \mathrm{diam}Y|.$$

From now on, we suppose that $\mathbb{R}^n$ is always endowed with the standard Euclidean norm.

Definition 2.7. *Denote by* $\mathrm{Iso}(\mathbb{R}^n)$ *the group of all isometries of* $\mathbb{R}^n$. *For arbitrary non-empty subsets* $X,Y \subset \mathbb{R}^n$*, the Euclidean Gromov-Hausdorff distance is*

$$\mathrm{d}_{EH}(X,Y) = \inf_{T \in \mathrm{Iso}(\mathbb{R}^n)} \mathrm{d}_H\big(X, T(Y)\big).$$

Theorem 2.1 ([6])**.** *Let* $X,Y \subset \mathbb{R}^n$ *be non-empty compact subsets. Then*

$$\mathrm{d}_{GH}(X,Y) \leq \mathrm{d}_{EH}(X,Y) \leq c_n' \cdot M^{\frac{1}{2}} \cdot \big(\mathrm{d}_{GH}(X,Y)\big)^{\frac{1}{2}},$$

where $M = \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}$ *and* c_n' *is a constant that depends only on* n.

Corollary 2.1. *Let* $X,Y \subset \mathbb{R}^n$ *be non-empty bounded subsets. Then*

$$\mathrm{d}_{GH}(X,Y) \leq \mathrm{d}_{EH}(X,Y) \leq c_n' \cdot M^{\frac{1}{2}} \cdot \big(\mathrm{d}_{GH}(X,Y)\big)^{\frac{1}{2}},$$

where $M = \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}$ *and* c_n' *is a constant that depends only on* n.

Proof. Since X and Y are bounded, $\overline{X}$ and $\overline{Y}$ are compact. Then the desired inequalities for X and Y follow from Theorem 2.1 and equalities

$$\begin{aligned} &\mathrm{d}_{GH}(X,Y) = \mathrm{d}_{GH}(\overline{X},\overline{Y}), && \mathrm{d}_{EH}(X,Y) = \mathrm{d}_{EH}(\overline{X},\overline{Y}), \\ &\mathrm{diam}\overline{X} = \mathrm{diam}X, && \mathrm{diam}\overline{Y} = \mathrm{diam}Y. \end{aligned}$$

The proof is complete. □

3 The main theorem

Now we formulate and prove the main theorem of this paper.

Theorem 3.1. *Let* $A \subset \mathbb{R}^n$, $t = \sup\{r : \exists B_r(x) \subset \mathbb{R}^n \setminus A\}$. *Then* $\mathrm{d}_{GH}(\mathbb{R}^n, A) < \infty$ *if and only if* $t < \infty$.

Proof. Suppose that $t<\infty$. Then A is a $(t+1)$-net in $\mathbb{R}^n$. Hence, $\mathrm{d}_H(\mathbb{R}^n,A)\le t+1<\infty$. It follows from the definition of Gromov-Hausdorff distance that

$$\mathrm{d}_{GH}(\mathbb{R}^n,A)\le \mathrm{d}_H(\mathbb{R}^n,A)\le t+1<\infty.$$

Suppose now that $\mathrm{d}_{GH}(\mathbb{R}^n,A)<\infty$. According to Claim 1, there exists a correspondence R between $\mathbb{R}^n$ and A with a distortion $\operatorname{dis}R=c<\infty$. Let us choose an arbitrary point $a\in A$ and some point $p\in R^{-1}(a)$.

Let c'_n be the constant from Corollary 2.1, and set $T=\sqrt{3c}c'_n$. Choose $N\in\mathbb{N}$ such that $3T\sqrt{N}<\frac{N}{2}$, $c<N$, $c<T\sqrt{N}$.

We set $X=S_N(p)$, $X'=X\cup\{p\}$, $Y=R(X)$, $Y'=Y\cup\{a\}$. Note that $Y\subset A$. The correspondence R, being restricted to X' and Y', generates a correspondence R' with distortion $\operatorname{dis}R'\le c$.

Note that $\operatorname{diam}X'=2N$. By Claim 4, $|\operatorname{diam}X'-\operatorname{diam}Y'|\le 2\mathrm{d}_{GH}(X',Y')$. By Claim 1, it holds $2\mathrm{d}_{GH}(X',Y')\le c$. Hence, $|\operatorname{diam}X'-\operatorname{diam}Y'|\le c$ and, thus, $\operatorname{diam}Y'\le 2N+c$. Therefore, X' and Y' are both bounded.

By Corollary 2.1, we obtain

$$\begin{aligned}\mathrm{d}_{EH}(X',Y')&\le c'_n\cdot\sqrt{\max\{\operatorname{diam}(X'),\operatorname{diam}(Y')\}}\cdot \mathrm{d}_{GH}(X',Y')^{\frac{1}{2}}\\&\le c'_n\cdot\sqrt{2N+c}\cdot\sqrt{c}<c'_n\sqrt{3N}\sqrt{c}=T\sqrt{N}.\end{aligned}$$

It follows from the definition of Euclidean Gromov-Hausdorff distance that there exists an isometry $f\colon\mathbb{R}^n\to\mathbb{R}^n$ such that $\mathrm{d}_H(X',f(Y'))<T\sqrt{N}$.

Now construct the relation $U\subset Y'\times X'$ in the following way:

$$U=\big\{(y,x):|f(y)x|<T\sqrt{N},y\in Y',x\in X'\big\}.$$

According to Claim 3, the inequality $\mathrm{d}_H(X',f(Y'))<T\sqrt{N}$ implies that U is a correspondence and $\operatorname{dis}U\le 2T\sqrt{N}$. Consider the relation $Q=U\circ R'\subset X'\times X'$. Since U and R' are correspondences, Q is a correspondence by Claim 2. By the same Claim 2, we have $\operatorname{dis}Q\le\operatorname{dis}U+\operatorname{dis}R'\le 2T\sqrt{N}+c$.

Choose an arbitrary $x\in Q^{-1}(p)$. Let us prove that $x=p$. Suppose $x\ne p$. Then $x\in X$. Take $x'\in X$ such that $|xx'|<T\sqrt{N}-c$ and choose some $q\in Q(x')$. By definition of distortion, $||pq|-|xx'||\le 2T\sqrt{N}+c$. Hence, $|pq|<3T\sqrt{N}<\frac{N}{2}<N$. Since $|pu|=N$ for every $u\in X$, we conclude that $q=p$. Similarly, $(p,x)\in Q$ implies that $x=p$. We have proved that $Q^{-1}(p)=Q(p)=\{p\}$. Since $R'(p)=\{a\}$, it follows that $U(a)=\{p\}$.

Define a cone $D=\cup_{t\in\mathbb{R}_+}tB_{\frac{N}{2}}(m)$ for some point $m\in S_N(0)$. Let us prove that for every cone T_D isometric to D with its vertex in $f(a)$, there exists $y\in Y$ such that

$f(y)\in T_D$. Consider a cone $T_D-f(a)+p$. Denote its axis by ℓ. Consider the point $w=\ell\cap X$. Since $\mathrm{d}_H(X',f(Y'))\le T\sqrt{N}$, there exists $y\in Y'$ such that $|wf(y)|\le T\sqrt{N}$. Since $U(a)=\{p\}$ and $|pw|=N>T\sqrt{N}$, it follows that $y\neq a$ and $y\in Y$.

Since $U(a)=\{p\}$ and $\mathrm{dis}(U)<T\sqrt{N}$, it follows that $|f(a)p|<T\sqrt{N}$. Since the point $w+f(a)-p$ belongs to the axis of the cone T_D, to show that $f(y)\in T_D$ it suffices to prove that $\|f(y)-w-f(a)+p\|\le \frac{N}{2}$. By triangle inequality,

$$\|f(y)-w-f(a)+p\|\le |f(y)w|+|f(a)p|<2T\sqrt{N}<\frac{N}{2}.$$

Hence, $f(y)\in T_D$. Since f is an isometry, it follows that for every cone T_D isometric to D with its vertex in a, there exists $y\in Y\cap T_D$. Since $\mathrm{dis} R'\le c$ and $Y=R(X)$, it follows that $||ya|-N|\le c$.

Therefore, we have proven the following statement: For an arbitrary point $a\in A$ and an arbitrary cone T_D isometric to D with its vertex in a, there exists a point $a'\in T_D\cap(B_{N+c}(a)\setminus B_{N-c}(a))\cap A$.

Suppose now that A is not an ε-net in $\mathbb{R}^n$ for each positive ε. Let us choose some ball $B_r(x)\subset\mathbb{R}^n\setminus A$. Without loss of generality, suppose that there exists a point $a\in A$ such that $a\in S_r(x)$. Consider a cone T_D isometric to D with its vertex at a, whose axis starts at a and contains x. Let us initially choose r so large that the following inclusion holds:

$$T_D\cap\big(B_{N+c}(a)\setminus B_{N-c}(a)\big)\subset B_r(x).$$

Therefore, according to the proven statement, we get $B_r(x)\cap A\neq\emptyset$, a contradiction. □

Theorem 3.1 cannot be generalized to arbitrary Euclidean normed spaces.

Example 3.1. Consider the space ℓ_2 of all sequences $(x_1,x_2,\dots)$ such that $\sum_{i=1}^{\infty}x_i^2<\infty$. It is isometric to its subspace

$$A=\big\{(x_1,x_2,\dots)\in\ell_2 : x_1=0\big\}.$$

However, A is not an ε-network in ℓ_2 for every $\varepsilon>0$ because

$$\mathrm{d}_H\big((\varepsilon,0,\dots),A\big)\ge\varepsilon.$$

Remark 3.1. As Example 3.1 shows, the finiteness of the dimension is a crucial condition in Theorem 3.1. In fact, it is required by [6, Theorem 1], on which Theorem 2.1 is based.

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