Commun. Math. Res.
doi: 10.4208/cmr.2024-0033

Vol. **41**, No. 1, pp. 9-24
February 2025

Degenerations of Nilalgebras

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Received 26 July 2024; Accepted 14 October 2024

Abstract. All complex 3-dimensional nilalgebras were described. As a corollary, all degenerations in the variety of complex 3-dimensional nilalgebras were obtained.

AMS subject classifications: Primary: 17A30; Secondary: 14L30

Key words: Nilalgebra, algebraic classification, geometric classification, degeneration.

1 Introduction

An element x is nil, if there exists a number n, such that for each $k \geq n$ we have $x^k = 0$[†]. An algebra is called a nilalgebra if each element is nil. The class of nilalgebras plays an important role in the ring theory. So, Köthe's problem is one of the old problems in ring and module theory that has not yet been solved. A problem of the existence of simple associative nil rings was actualized by Kaplansky and successfully solved by Smoktunowicz [15]. Another famous problem was posted by Albert: Is every finite-dimensional (commutative) power associative nilalgebra solvable? It is still open, but it was solved in some particular cases [14]. In the present note, we give a positive answer to the problem of Albert for non-anticommutative 3-dimensional algebras. Let us note, that in the anticommutative case, the problem of Albert does not make sense: each anticommutative algebra is nilalgebra with nilindex 2 and in almost all dimensions there are simple Lie algebras, that are not solvable. On the other hand, for each $n > 3$, Correa

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†By x^k we mean all possible arrangements of non-associative products.

and Hentzel constructed a non-(anti)commutative n-dimensional non-solvable nilalgebra [3]. To do it, we obtain the full classification of complex 3-dimensional nilalgebras and as a result, we have a geometric classification and the description of all degenerations in the variety of complex 3-dimensional nilalgebras. In particular, we proved that this variety of algebras has two rigid algebras. Let us note, that the classification of 4-dimensional commutative nilalgebras is given in [4]. It is known that each right Leibniz algebra (i.e., an algebra satisfying the identity $x(yz)=(xy)z+y(xz)$; about Leibniz algebras see, for example, [11] and references therein) satisfies the identity $x^2x=0$. Hence, the variety of symmetric Leibniz algebras (i.e., left and right Leibniz algebras) gives a subvariety in the variety of nilalgebras. Thanks to [1], the intersection of right mono Leibniz (i.e. algebras where each one-generated subalgebra is a right Leibniz algebra) and left mono Leibniz algebras gives the variety of nilalgebras with nilindex 3. As one more corollary from our result, we have the algebraic and geometric classification of 3-dimensional symmetric mono Leibniz algebras.

2 The algebraic classification of 3-dimensional nilalgebras

2.1 Nilalgebras with nilindex 3

By identity $x^3=0$ we mean the system of two identities

$$x^2x=0 \quad \text{and} \quad xx^2=0.$$

Linearizing them, we obtain a pair of useful identities

$$(xy+yx)x=-x^2y \quad \text{and} \quad x(xy+yx)=-yx^2. \tag{2.1}$$

The full linearization gives the following two identities:

$$\sum_{\sigma\in S_3}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}=0 \quad \text{and} \quad \sum_{\sigma\in S_3}x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})=0.$$

We aim to classify all complex 3-dimensional algebras A satisfying $x^3=0$. Obviously, each anticommutative algebra has this property. These algebras were classified in [7]. Hence, we will consider only non-anticommutative cases, i.e. algebras where there is an element a, such that $a^2\neq 0$. It is easy to see, that a and a^2 are linearly independent and (2.1) gives $a^2a^2=0$. Consider a basis $\{a,a^2,b\}$ of the algebra A, then we have the following statement.