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Degenerations of Nilalgebras

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Abstract. All complex 3-dimensional nilalgebras were described. As a corollary, all degenerations in the variety of complex 3-dimensional nilalgebras were obtained.

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1 Introduction

An element x is nil, if there exists a number n, such that for each $k \geq n$ we have $x^k = 0$[†]. An algebra is called a nilalgebra if each element is nil. The class of nilalgebras plays an important role in the ring theory. So, Köthe's problem is one of the old problems in ring and module theory that has not yet been solved. A problem of the existence of simple associative nil rings was actualized by Kaplansky and successfully solved by Smoktunowicz [15]. Another famous problem was posted by Albert: Is every finite-dimensional (commutative) power associative nilalgebra solvable? It is still open, but it was solved in some particular cases [14]. In the present note, we give a positive answer to the problem of Albert for non-anticommutative 3-dimensional algebras. Let us note, that in the anticommutative case, the problem of Albert does not make sense: each anticommutative algebra is nilalgebra with nilindex 2 and in almost all dimensions there are simple Lie algebras, that are not solvable. On the other hand, for each $n > 3$, Correa

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†By x^k we mean all possible arrangements of non-associative products.

and Hentzel constructed a non-(anti)commutative n-dimensional non-solvable nilalgebra [3]. To do it, we obtain the full classification of complex 3-dimensional nilalgebras and as a result, we have a geometric classification and the description of all degenerations in the variety of complex 3-dimensional nilalgebras. In particular, we proved that this variety of algebras has two rigid algebras. Let us note, that the classification of 4-dimensional commutative nilalgebras is given in [4]. It is known that each right Leibniz algebra (i.e., an algebra satisfying the identity $x(yz)=(xy)z+y(xz)$; about Leibniz algebras see, for example, [11] and references therein) satisfies the identity $x^2x=0$. Hence, the variety of symmetric Leibniz algebras (i.e., left and right Leibniz algebras) gives a subvariety in the variety of nilalgebras. Thanks to [1], the intersection of right mono Leibniz (i.e. algebras where each one-generated subalgebra is a right Leibniz algebra) and left mono Leibniz algebras gives the variety of nilalgebras with nilindex 3. As one more corollary from our result, we have the algebraic and geometric classification of 3-dimensional symmetric mono Leibniz algebras.

2 The algebraic classification of 3-dimensional nilalgebras

2.1 Nilalgebras with nilindex 3

By identity $x^3=0$ we mean the system of two identities

$$x^2x=0 \quad \text{and} \quad xx^2=0.$$

Linearizing them, we obtain a pair of useful identities

$$(xy+yx)x=-x^2y \quad \text{and} \quad x(xy+yx)=-yx^2. \tag{2.1}$$

The full linearization gives the following two identities:

$$\sum_{\sigma\in S_3}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}=0 \quad \text{and} \quad \sum_{\sigma\in S_3}x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})=0.$$

We aim to classify all complex 3-dimensional algebras A satisfying $x^3=0$. Obviously, each anticommutative algebra has this property. These algebras were classified in [7]. Hence, we will consider only non-anticommutative cases, i.e. algebras where there is an element a, such that $a^2\neq 0$. It is easy to see, that a and a^2 are linearly independent and (2.1) gives $a^2a^2=0$. Consider a basis $\{a,a^2,b\}$ of the algebra A, then we have the following statement.

Lemma 2.1. *For the algebra* A *with a basis* $\{a,a^2,b\}$ *there are the following three cases:*

$$(T_0)\,:\,ba=0,\quad (T_a)\,:\,ba=a,\quad (T_b)\,:\,ba=b.$$

Proof. Let $ba=\lambda a+\mu a^2+\nu b$. Suppose first that $\nu\neq 0$. By using the substitution $\frac{a}{\nu}\mapsto a$, we can suppose that $\nu=1$. Let

$$b'=b+\lambda a+(\lambda+\mu)a^2,$$

then

$$b'a=\big(b+\lambda a+(\lambda+\mu)a^2\big)a=\lambda a+\mu a^2+b+\lambda a^2=b'.$$

Thus, for $\nu\neq 0$ the algebra A by substitution $b'\mapsto b$ is of type (T_b). Let now $\nu=0$. Denote $b'=-\mu a+b$, then

$$b'a=-\mu a^2+\lambda a+\mu a^2=\lambda a.$$

Thus, for $\lambda=0$ the algebra A by the substitution $b'\mapsto b$ is of type (T_0), and for $\lambda\neq 0$ by the substitution $\frac{b'}{\lambda}\mapsto b$ to type (T_a). □

2.1.1 Commutative nilalgebras with nilindex 3

Lemma 2.2. *If the algebra* A *is commutative, then in some basis* $\{a,a^2,b\}$ *it is determined by one of the multiplication tables*

$$\begin{aligned} C_{0,0}\ &:\ ab=a^2b=0,\quad b^2=0,\\ C_{0,a^2}\ &:\ ab=a^2b=0,\quad b^2=a^2. \end{aligned}$$

Proof. Using the identity (2.1) and commutativity, we have

$$a^2b=-2(ba)a,\quad b^2a=-2(ab)b.$$

Hence, we have the following relations in the three cases considered:

$$(T_0)\,:\,\begin{cases} a^2b=0,\\ b^2a=0,\end{cases}\qquad (T_a)\,:\,\begin{cases} a^2b=-2a^2,\\ b^2a=-2a,\end{cases}\qquad (T_b)\,:\,\begin{cases} a^2b=-2b,\\ b^2a=-2b^2.\end{cases}$$

It remains to determine b^2 in each of the three cases. Let us suppose that $b^2=\lambda a+\mu a^2+\nu b$.

Case (T_0)**.** First, $b^2a=0$. On the other hand, $b^2a=\lambda a^2$. Therefore, $\lambda=0$ and $b^2=\mu a^2+\nu b$. Further, $b^2b=0$ and $b^2b=\nu b^2=\nu\mu a^2+\nu^2 b$. So $\nu=0$ and $b^2=\mu a^2$. Thus, we get two cases: either $b^2=0$, or (after substituting $\frac{b}{\sqrt{\mu}}\mapsto b$) $b^2=a^2$. So, in this case we have two algebras: $C_{0,0}$ and C_{0,a^2}.

Case (T_a)**.** First note that $b^2a=-2a$. On the other hand, $b^2a=\lambda a^2+\nu a$. So $b^2=\mu a^2-2b$. Further, $0=b^2b=\mu a^2b-2b^2=-2\mu a^2-2\mu a^2+4b$. Thus, the case (T_a) is not realized in commutative algebras.

Case (T_b)**.** We have $b^2a=\lambda a^2+\nu b$ and $b^2a=-2(\lambda a+\mu a^2+\nu b)$. Then $\lambda=2\mu=3\nu=0$, i.e. $b^2=0$. Further, if $x=a^2+b$, then $x^2=2a^2b=-4b$, i.e. $0=x^2x=8b$. Thus, the case (T_b) is neither realized in commutative algebras.

It is easy to check that $\dim \mathrm{Ann} C_{0,0}=2$ and $\dim \mathrm{Ann} C_{0,a^2}=1$. Hence, they are non-isomorphic. □

2.1.2 Noncommutative nilalgebras with nilindex 3

Let us consider the algebra A^+ with the multiplication $x\cdot y=\frac{1}{2}(xy+yx)$. It is easy to see, that A^+ is commutative and satisfies the identities $x^3=0$.

Lemma 2.3. *If* A *be a noncommutative nilalgebra with nilindex 3 and* A^+ *has type* $C_{0,0}$ *or* C_{0,a^2} *regarding a basis* $\{a,a^2,b\}$*, then it satisfies:*

(1) $b^2=0$ *or* $b^2=a^2$,

(2) $ba=-ab$,

(3) $a^2a^2=a^2b=ba^2=0$.

Proof. It is easy to see, that $a\cdot a=a^2$ and $b^2=b\cdot b=c$, where $c=0$ or $c=a^2$. From $a\cdot b=0$ follows $ab=-ba$. The relation (2.1) gives $a^2b=-2(a\cdot b)a=0$ and on the other hand, we have $a^2b+ba^2=2(a\cdot a)\cdot b=0$, then $ba^2=-a^2b=0$. In the end, we obtain $a^2a^2=(a\cdot a)\cdot(a\cdot a)=0$. □

To summarize, for a description of the multiplication table of A in the basis $\{a,a^2,b\}$, we have to determine the value of ab. Let $ab=\lambda a+\mu a^2+\nu b$.

Let us define by $N_{c,d}$ a 3-dimensional algebra with a basis $\{a,a^2,b\}$ and the multiplication table given below

$$aa=a^2,\quad ab=-ba=c,\quad b^2=d.$$

Proposition 2.1. *If* $\mathrm{A}^+\cong C_{0,0}$*, then* A *is isomorphic to* $C_{0,0}$*,* $N_{a,0}$*,* $N_{b,0}$*, or* $N_{a^2,0}$.

Proof. Let A be an algebra with a basis $\{a,a^2,b\}$ satisfying the following relation:

$$-ba=ab=\lambda a+\mu a^2+\nu b,$$

and A' be an algebra with the basis $\{a',a'^2,b'\}$ and the following relation:

$$-b'a'=a'b'=\lambda'a'+\mu'a'^2+\nu'b'.$$

We suppose, that $\mathrm{A}'\cong\mathrm{A}$ and let

$$\begin{aligned}\xi(a')&=\alpha_1a+\beta_1a^2+\gamma_1b,\\ \xi(b')&=\alpha_2a+\beta_2a^2+\gamma_2b.\end{aligned}$$

Then,

$$0=\xi(0)=\xi(b'^2)=(\xi(b'))^2=\alpha_2^2a^2\quad\text{and}\quad\alpha_2=0.$$

By a similar way, $\xi(a'^2)=(\xi(a'))^2=\alpha_1^2a^2$.

It is easy to see that

$$\begin{aligned}\xi(a')\xi(b')&=(\alpha_1a+\beta_1a^2+\gamma_1b)(\beta_2a^2+\gamma_2b)\\ &=\alpha_1\gamma_2ab=\alpha_1\gamma_2(\lambda a+\mu a^2+\nu b).\end{aligned}$$

On the other hand,

$$\begin{aligned}\xi(a'b')&=\xi(\lambda'a'+\mu'a'^2+\nu'b')\\ &=\lambda'(\alpha_1a+\beta_1a^2+\gamma_1b)+\mu'\alpha_1^2a^2+\nu'(\beta_2a^2+\gamma_2b).\end{aligned}$$

The last two relations give the following system of equalities:

$$\begin{aligned}&\alpha_1\gamma_2\lambda=\lambda'\alpha_1,\\ &\alpha_1\gamma_2\mu=\lambda'\beta_1+\mu'\alpha_1^2+\nu'\beta_2,\\ &\alpha_1\gamma_2\nu=\lambda'\gamma_1+\nu'\gamma_2.\end{aligned}$$

Since, $\alpha_2=0$ and $\dim\mathrm{Im}\xi=3$, then $\alpha_1\gamma_2\neq0$, otherwise $\xi(a'^2),\xi(b')\in\langle a^2\rangle$. Then we have

$$\lambda'=\gamma_2\lambda,\quad \mu'\alpha_1^2=\gamma_2(\alpha_1\mu-\beta_1\lambda)-\beta_2(\alpha_1\nu-\gamma_1\lambda),\quad \nu'=\alpha_1\nu-\gamma_1\lambda.$$

(1) If $\lambda\neq0$, then by choosing $\gamma_2=\frac{1}{\lambda}$, $\gamma_1=\frac{\alpha_1\nu}{\lambda}$, and $\beta_1=\frac{\alpha_1\mu}{\lambda}-\frac{\beta_2}{\gamma_2\lambda}(\alpha_1\nu-\gamma_1\lambda)$, we have $\lambda'=1$, $\nu'=0$, and $\mu'=0$. Hence, we have the algebra $N_{a',0}$ with the basis $\{a',a'^2,b'\}$.

(2) If $\lambda=0$, then

$$\lambda'=0,\quad \mu'\alpha_1=\gamma_2\mu-\beta_2\nu,\quad \nu'=\alpha_1\nu.$$

(a) If $\nu \neq 0$, then by choosing $\alpha_1 = \frac{1}{\nu}$ and $\beta_2 = \frac{\gamma_2 \mu}{\nu}$, we have $\nu' = 1$ and $\mu' = 0$. This case gives the algebra $N_{b',0}$.

(b) If $\nu = 0$, then
$$\lambda' = 0, \quad \mu' \alpha_1 = \gamma_2 \mu, \quad \nu' = 0.$$
If $\mu = 0$ we have the algebra $C_{0,0}$; if $\mu \neq 0$ and $\alpha_1 = \gamma_2 \mu$, we have $\mu' = 1$. The last gives the algebra $N_{a'^2,0}$.

The proof is complete. □

Proposition 2.2. *If* $\mathrm{A}^+ \cong C_{0,a^2}$, *then* A *is isomorphic to* N_{b,a^2} *or* $N_{\alpha a^2, a^2}$, *for an element* $\alpha \in \mathbb{C}$.

Proof. We will follow the ideas from the previous statement for $b^2 = a^2$. Hence,
$$\begin{aligned} -ba &= ab = \lambda a + \mu a^2 + \nu b, & b^2 &= a^2, \\ -b'a' &= a'b' = \lambda' a' + \mu' a'^2 + \nu' b', & b'^2 &= a'^2, \\ \xi(a') &= \alpha_1 a + \beta_1 a^2 + \gamma_1 b, \\ \xi(b') &= \alpha_2 a + \beta_2 a^2 + \gamma_2 b. \end{aligned}$$

Then,
$$\begin{aligned} \xi(b'^2) &= \big(\xi(b')\big)^2 = \big(\alpha_2^2 + \gamma_2^2\big) a^2, \\ \xi(a'^2) &= \big(\xi(a')\big)^2 = \big(\alpha_1^2 + \gamma_1^2\big) a^2. \end{aligned}$$

Let us note that $b'^2 = a'^2 \neq 0$, then $\alpha_2^2 + \gamma_2^2 = \alpha_1^2 + \gamma_1^2 \neq 0$. It follows,
$$\begin{aligned} \xi(a')\xi(b') &= (\alpha_1 a + \beta_1 a^2 + \gamma_1 b)(\alpha_2 a + \beta_2 a^2 + \gamma_2 b) \\ &= (\alpha_1\alpha_2 + \gamma_1\gamma_2) a^2 + (\alpha_1\gamma_2 - \alpha_2\gamma_1) ab \\ &= (\alpha_1\alpha_2 + \gamma_1\gamma_2) a^2 + (\alpha_1\gamma_2 - \alpha_2\gamma_1)(\lambda a + \mu a^2 + \nu b), \\ \xi(a'b') &= \xi(\lambda' a' + \mu' a'^2 + \nu' b') \\ &= \lambda'(\alpha_1 a + \beta_1 a^2 + \gamma_1 b) + \mu'(\alpha_1^2 + \gamma_1^2) a^2 + \nu'(\alpha_2 a + \beta_2 a^2 + \gamma_2 b). \end{aligned}$$

The last two relations give the following system of equalities:
$$\begin{aligned} &(\alpha_1\gamma_2 - \alpha_2\gamma_1)\lambda = \lambda'\alpha_1 + \nu'\alpha_2, \\ &\alpha_1\alpha_2 + \gamma_1\gamma_2 + (\alpha_1\gamma_2 - \alpha_2\gamma_1)\mu = \lambda'\beta_1 + \mu'\big(\alpha_1^2 + \gamma_1^2\big) + \nu'\beta_2, \\ &(\alpha_1\gamma_2 - \alpha_2\gamma_1)\nu = \lambda'\gamma_1 + \nu'\gamma_2. \end{aligned}$$

Since elements from a basis are linearly independent, we have that $\Delta=\alpha_1\gamma_2-\alpha_2\gamma_1\neq 0$, otherwise, there are $k_1,k_2\in\mathbb{C}$ such that $\xi(k_1a'+k_2b'),\xi(a'^2)\in\langle a^2\rangle$. Hence, we have the following relations, that we denote as $(\star)$:

$$\begin{aligned}&\lambda'=\gamma_2\lambda-\alpha_2\nu,\\&\Delta\mu=\lambda'\beta_1+\nu'\beta_2+\mu'\left(\alpha_1^2+\gamma_1^2\right)-(\alpha_1\alpha_2+\gamma_1\gamma_2),\\&\nu'=\alpha_1\nu-\gamma_1\lambda.\end{aligned}$$

(1) If $\lambda=\nu=0$, then $\lambda'=\nu'=0$. Hence, for $\mu'=0$ we have the commutative algebra C_{0,a^2}, for $\mu'\neq 0$ — noncommutative algebras $N_{\alpha a^2,a^2}$, where $\alpha\neq 0$. We can joint these cases in one family $N_{\alpha a^2,a^2}$.

(2) If $\lambda\neq 0$, then by choosing $\alpha_2=1$, $\gamma_2=\frac{\nu}{\lambda}$, we have $\lambda'=0$. Hence,

$$\lambda'=0,\quad \Delta\mu=\mu'\left(\alpha_1^2+\gamma_1^2\right)+\nu'\beta_2-(\alpha_1\alpha_2+\gamma_1\gamma_2),\quad \nu'=-\gamma_1\lambda+\alpha_1\nu.$$

Since, $\Delta\lambda=(\alpha_1\gamma_2-\alpha_2\gamma_1)\lambda=\alpha_1\nu-\gamma_1\lambda$, then $\nu'=\Delta\lambda\neq 0$. If $\beta_2=\frac{\Delta\mu+(\alpha_1\alpha_2+\gamma_1\gamma_2)}{\Delta\lambda}$, then $\mu'=0$. Hence, in this case N is isomorphic to one algebra from the family $N_{\alpha b',a'^2}$. It is clear that $\alpha\neq 0$ and for different $\alpha\neq 0$, these algebras are isomorphic. Next, for $a''=\frac{a'}{\alpha}$ and $b''=\frac{b'}{\alpha}$, we have N_{b'',a''^2}.

(3) By symmetry on a and b in relations $(\star)$, we have to consider only the case $\lambda=0,\nu\neq 0$. Hence, be choosing some suitable nonzero α_1 and α_2, we obtain the previous case.

The proof is complete. □

Proposition 2.3. *Algebras $C_{0,0}$, $N_{a,0}$, $N_{b,0}$, $N_{a^2,0}$, N_{b,a^2} and $N_{\alpha a^2,a^2}$ are non-isomorphic, except $N_{\alpha a^2,a^2}\cong N_{-\alpha a^2,a^2}$.*

Proof. First, commutative algebras are not isomorphic to noncommutative. Second, if $\mathrm{A}^+\ncong\mathrm{B}^+$, then $\mathrm{A}\ncong\mathrm{B}$. Third, $N_{a,0}^2=\langle a,a^2\rangle$, $N_{b,0}^2=\langle a^2,b\rangle$, $N_{a^2,0}^2=\langle a^2\rangle$. Hence, $N_{a^2,0}\ncong N_{a,0}$ and $N_{a^2,0}\ncong N_{b,0}$. Since, $(N_{a,0}^2)^2\neq 0$ and $(N_{b,0}^2)^2=0$, we have $N_{a,0}\ncong N_{b,0}$. Similarly, $N_{\alpha a^2,a^2}\ncong N_{b,a^2}$.

Let us consider two isomorphic algebras $N=N_{\lambda a^2,a^2}$ and $N'=N_{\mu a'^2,a'^2}$. Let ξ be an isomorphism between them such that

$$\begin{aligned}&\xi(a')=\alpha_1a+\beta_1a^2+\gamma_1b,\\&\xi(b')=\alpha_2a+\beta_2a^2+\gamma_2b.\end{aligned}$$

It follows that $\alpha_2^2+\gamma_2^2=\alpha_1^2+\gamma_1^2\neq 0$. Hence,

$$\begin{aligned}\xi(a')\xi(b')&=(\alpha_1 a+\beta_1 a^2+\gamma_1 b)(\alpha_2 a+\beta_2 a^2+\gamma_2 b)\\&=\big(\alpha_1\alpha_2+\gamma_1\gamma_2+\lambda(\alpha_1\gamma_2-\alpha_2\gamma_1)\big)a^2,\\\xi(b')\xi(a')&=\big(\alpha_1\alpha_2+\gamma_1\gamma_2-\lambda(\alpha_1\gamma_2-\alpha_2\gamma_1)\big)a^2,\\\xi(a'b')&=\xi\big(\mu(a')^2\big)=\mu\big(\alpha_1^2+\gamma_1^2\big)a^2.\end{aligned}$$

The last relations give the following system of equalities:

$$\alpha_1\alpha_2+\gamma_1\gamma_2=0,\quad \lambda(\alpha_1\gamma_2-\alpha_2\gamma_1)=\mu\big(\alpha_1^2+\gamma_1^2\big),\quad \alpha_2^2+\gamma_2^2=\alpha_1^2+\gamma_1^2\neq 0.$$

(1) If $\alpha_1=0$, then $\gamma_1\neq 0$ and $\gamma_2=0$. It follows that $\alpha_2=\pm\gamma_1$ and $\mu=\pm\lambda$.

(2) If $\alpha_1\neq 0$, then $\alpha_2=-\frac{\gamma_1\gamma_2}{\alpha_1}$ and $\frac{\gamma_2^2}{\alpha_1^2}(\gamma_1^2+\alpha_1^2)=\alpha_1^2+\gamma_1^2$. That gives $\alpha_1^2=\gamma_2^2$ and $\alpha_2^2=\gamma_1^2$. Hence, $\alpha_1=\pm\gamma_2$ and $\alpha_2=\mp\gamma_1$, i.e. $\mu=\pm\lambda$.

The proof is complete. □

2.2 Nilalgebras with nilindex 4

Let us now consider complex 3-dimensional nilalgebras with nilindex 4. It means, that the algebra A has an element a, such that $a^2\neq 0$; at least one element from aa^2 and a^2a is nonzero; and $a^k=0$ for each $k>3$. Let us suppose that $a^2a\neq 0$. If $a^2a\in\langle a,a^2\rangle$, then $a^2a=\alpha a+\beta a^2$ and

$$0=(a^2a)a=\alpha a^2+\beta a^2a=\alpha\beta a+(\alpha+\beta^2)a^2,$$

i.e. $\alpha=\beta=0$.

We can choose the basis $\{a,a^2,a^2a\}$ and define the multiplication on this algebra. Let $aa^2=\gamma_1 a+\gamma_2 a^2+\gamma_3 a^2a$, then $0=(aa^2)a=\gamma_1 a^2+\gamma_2 a^2a$ and $\gamma_1=\gamma_2=0$. It is easy to see that A is nilpotent. The case $a^2a=0$ and $aa^2\neq 0$ is similar. The classification of complex 3-dimensional nilpotent algebras is given in [6]. Hence, A is isomorphic to one of the following algebras:

$$\begin{aligned}\mathrm{N}_1 &: e_1e_1=e_2\quad e_2e_1=e_3,\\\mathrm{N}_2^{\alpha} &: e_1e_1=e_2\quad e_1e_2=e_3\quad e_2e_1=\alpha e_3.\end{aligned}$$

2.3 Nilalgebras with nilindex 5

Let us now consider complex 3-dimensional nilalgebras with nilindex 5. It means, that the algebra A has an element a, such that $a^2 \neq 0$; at least one from elements a^2a^2, $(aa^2)a$, $a(aa^2)$, $(a^2a)a$ and $a(a^2a)$ is nonzero; and $a^k = 0$ for each $k > 4$.

(1) Let us suppose that $a^2a^2 \neq 0$. Following the same idea as in the Subsection 2.2, we have that $a^2a^2 \notin \langle a, a^2 \rangle$, then we can choose the basis $\{a, a^2, a^2a^2\}$ and define the multiplication on this algebra. It will be a nilpotent algebra.

(2) If $(aa^2)a \neq 0$, then $aa^2 \neq 0$, and we can choose the basis $\{a, a^2, aa^2\}$. Hence, $(aa^2)a = \alpha a + \beta a^2 + \gamma aa^2$, then

$$\begin{aligned} 0 &= \big(((aa^2)a)a^2\big)a = \alpha(aa^2)a, \\ 0 &= \big(a((aa^2)a)\big)a = \alpha a^2 a + \beta(aa^2)a, \\ 0 &= \big((aa^2)a\big)a = \alpha a^2 + \beta a^2 a + \gamma(aa^2)a. \end{aligned}$$

The last gives $\alpha = \beta = \gamma = 0$ and this case can not be realized.

(3) If one of $a(aa^2)$, $(a^2a)a$ or $a(a^2a)$ is not equal to zero, we will apply a similar idea and can obtain that the case can not be realized.

The classification of complex 3-dimensional nilpotent algebras is given in [6]. Hence, A is isomorphic to one of the following algebras:

$$\begin{aligned} \mathbf{N_1} &: e_1e_1 = e_2 \quad e_2e_2 = e_3, \\ \mathbf{N_2} &: e_1e_1 = e_2 \quad e_2e_1 = e_3 \quad e_2e_2 = e_3. \end{aligned}$$

2.4 Nilalgebras with nilindex $k > 5$

Let us now consider complex 3-dimensional nilalgebras with nilindex $k > 5$. It means, that algebra A has an element a, such that $a^k \neq 0$. It means, that there is an arrangement of brackets in the non-associative word a^k such that the result is nonzero. For this nonzero arrangement of brackets, we can write a^k as one of the following forms:

$$\begin{aligned} a^k &= \big((aa)(aa)\big) T_{a^{k_1}} \dots T_{a^{k_m}} \quad \text{with} \quad k_1 + \dots + k_m + 4 = k, \\ a^k &= \big((aa)a\big) T_{a^{k_1}} \dots T_{a^{k_m}} \quad \text{with} \quad k_1 + \dots + k_m + 3 = k, \end{aligned}$$

or

$$a^k = \big(a(aa)\big)\, T_{a^{k_1}} \dots T_{a^{k_m}} \quad \text{with} \quad k_1 + \cdots + k_m + 3 = k,$$

where T_x is a left or right multiplication on the element x. Following the idea from the previous subsection case (2), we can choose a basis of A as $\{a, a^2, Q\}$, where $Q \in \{(aa)(aa), (aa)a, a(aa)\}$ and applying the similar arguments, we obtain that the present case can not be realized. Since there are no 3-dimensional nilalgebras with nilindex $k > 5$.

2.5 The classification theorem

The classification of 3-dimensional nilalgebras with nilindex 2 (=anticommutative algebras) is given in [7]. The classification of 3-dimensional nilalgebras with nilindex 3, 4, and 5 is given in the previous subsections. Hence, we are ready to summarize these results in the following theorem.

Theorem 2.1. *Let* N *be a complex 3-dimensional nilalgebra. Then* N *is isomorphic to an algebra from the following list:*

$$
\begin{array}{lllll}
\mathfrak{g}_1 & : e_2e_3 = e_1 & e_3e_2 = -e_1, & & \\
\mathfrak{g}_2 & : e_1e_3 = e_1 & e_2e_3 = e_2 & e_3e_1 = -e_1 & e_3e_2 = -e_2, \\
\mathfrak{g}_3^\alpha & : e_1e_3 = e_1 + e_2 & e_2e_3 = \alpha e_2 & e_3e_1 = -e_1 - e_2 & e_3e_2 = -\alpha e_2, \\
\mathfrak{g}_4 & : e_1e_2 = e_3 & e_1e_3 = -e_2 & e_2e_3 = e_1 & \\
 & e_2e_1 = -e_3 & e_3e_1 = e_2 & e_3e_2 = -e_1, & \\
\mathcal{A}_1^\alpha & : e_1e_2 = e_3 & e_1e_3 = e_1 + e_3 & e_2e_3 = \alpha e_2 & \\
 & e_2e_1 = -e_3 & e_3e_1 = -e_1 - e_3 & e_3e_2 = -\alpha e_2, & \\
\mathcal{A}_2 & : e_1e_2 = e_1 & e_2e_3 = e_2 & e_2e_1 = -e_1 & e_3e_2 = -e_2, \\
\mathcal{A}_3 & : e_1e_2 = e_3 & e_1e_3 = e_1 & e_2e_3 = e_2 & \\
 & e_2e_1 = -e_3 & e_3e_1 = -e_1 & e_3e_2 = -e_2, & \\
\mathcal{N}_1 & : e_1e_1 = e_2, & & & \\
\mathcal{N}_2 & : e_1e_1 = e_2 & e_1e_3 = e_1 & e_3e_1 = -e_1, & \\
\mathcal{N}_3 & : e_1e_1 = e_2 & e_1e_3 = e_3 & e_3e_1 = -e_3, & \\
\mathcal{N}_4 & : e_1e_1 = e_2 & e_1e_3 = e_2 & e_3e_1 = -e_2, & \\
\mathcal{N}_5 & : e_1e_1 = e_2 & e_1e_3 = e_3 & e_3e_1 = -e_3 & e_3e_3 = e_2, \\
\mathcal{N}_6^\alpha & : e_1e_1 = e_2 & e_1e_3 = \alpha e_2 & e_3e_1 = -\alpha e_2 & e_3e_3 = e_2,
\end{array}
$$

$$\begin{array}{llll}
\mathrm{N}_1 & : e_1e_1=e_2 & e_2e_1=e_3, & \\
\mathrm{N}_2^\alpha & : e_1e_1=e_2 & e_1e_2=e_3 & e_2e_1=\alpha e_3, \\
\mathbf{N_1} & : e_1e_1=e_2 & e_2e_2=e_3, & \\
\mathbf{N_2} & : e_1e_1=e_2 & e_2e_1=e_3 & e_2e_2=e_3.
\end{array}$$

All algebras are non-isomorphic, except $\mathfrak{g}_3^\alpha \cong \mathfrak{g}_3^{\alpha^{-1}}$, $\mathcal{A}_1^\alpha \cong \mathcal{A}_1^{\alpha^{-1}}$, *and* $\mathcal{N}_6^\alpha \cong \mathcal{N}_6^{-\alpha}$.

Let us recall the Albert's problem: Is every finite-dimensional (commutative) power associative nilalgebra solvable? For each $n>3$, Correa and Hentzel [3] constructed a non-(anti)commutative n-dimensional non-solvable nilalgebra. It is easy to see, that if a 2-dimensional algebra is a nilalgebra, then it should be commutative or anticommutative. Hence, Theorem 2.1 gives the following corollary.

Corollary 2.1. *Albert's problem is true for all non-anticommutative 3-dimensional algebras.*

Remark 2.1. The famous Nagata-Higman-Dubnov-Ivanov's theorem says that each associative nilalgebra is nilpotent. It is easy to see, that the algebra $\mathcal{N}_3$ is a non-nilpotent nilalgebra, that satisfies identities of the following type:

$$\begin{aligned}
&\alpha_1(xy)z+\alpha_2(yx)z+\alpha_3(xz)y+\alpha_4(zy)x+\alpha_5(yz)x+\alpha_6(zx)y \\
&\quad+\alpha_7z(xy)+\alpha_8z(yx)+\alpha_9y(xz)+\alpha_{10}x(zy) \\
&\quad+(-\alpha_1+\alpha_2+\alpha_7-\alpha_8-\alpha_4+\alpha_5+\alpha_{10})x(yz) \\
&\quad+(-\alpha_1+\alpha_2+\alpha_7-\alpha_8-\alpha_3+\alpha_6+\alpha_9)y(zx)=0.
\end{aligned}$$

Hence, each variety defined by an identity of the type given above does not have an analog of Nagata-Higman-Dubnov-Ivanov's theorem. In particular, the following identities have the above-given type:

(1) (Right) Leibniz: $(xy)z=(xz)y+x(yz)$.

(2) Reverse (right) Leibniz: $(xy)z=(zy)x+y(zx)$.

(3) Weakly associative: $(xy)z-x(yz)+(yz)x-y(zx)-(yx)z+y(xz)=0$.

(4) 2-step Jordan nilpotent: $(x\cdot y)\cdot z=(xy)z+(yx)z+z(xy)+z(yx)=0$.

(5) Almost anticommutative: $(xy)z+(yx)z=0$.

Thanks to [1], the intersection of right mono Leibniz (i.e. algebras where each one-generated subalgebra is a right Leibniz algebra) and left mono Leibniz algebras gives the variety of nilalgebras with nilindex 3. Hence, we have the following corollary.

Corollary 2.2. *The algebraic classification of symmetric mono Leibniz algebras is given in Theorem* 2.1. *Namely, it consists from algebras of $\mathfrak{g}_i$, $\mathcal{A}_i$, or $\mathcal{N}_i$ type.*

3 Degenerations of 3-dimensional nilalgebras

The study of varieties of non-associative algebras from a geometric point of view has a long story (see, [5–10, 12] and references therein). The geometric classification of algebras from a certain variety is based on the notion of degeneration, that is a "dual" notion to deformations [2, 13].

3.1 Definitions and notation

Given an n-dimensional vector space $\mathbb{V}$, the set $\mathrm{Hom}(\mathbb{V}\otimes\mathbb{V},\mathbb{V})\cong\mathbb{V}^*\otimes\mathbb{V}^*\otimes\mathbb{V}$ is a vector space of dimension n^3. This space has the structure of the affine variety $\mathbb{C}^{n^3}$. Indeed, let us fix a basis $e_1,\ldots,e_n$ of $\mathbb{V}$. Then any $\mu\in\mathrm{Hom}(\mathbb{V}\otimes\mathbb{V},\mathbb{V})$ is determined by n^3 structure constants $c_{ij}^k\in\mathbb{C}$ such that $\mu(e_i\otimes e_j)=\sum_{k=1}^n c_{ij}^k e_k$. A subset of $\mathrm{Hom}(\mathbb{V}\otimes\mathbb{V},\mathbb{V})$ is Zariski-closed if it can be defined by a set of polynomial equations in the variables c_{ij}^k $(1\le i,j,k\le n)$.

Let T be a set of polynomial identities. The set of algebra structures on $\mathbb{V}$ satisfying polynomial identities from T forms a Zariski-closed subset of the variety $\mathrm{Hom}(\mathbb{V}\otimes\mathbb{V},\mathbb{V})$. We denote this subset by $\mathbb{L}(T)$. The general linear group $\mathrm{GL}(\mathbb{V})$ acts on $\mathbb{L}(T)$ by conjugations

$$(g*\mu)(x\otimes y)=g\mu(g^{-1}x\otimes g^{-1}y)$$

for $x,y\in\mathbb{V}$, $\mu\in\mathbb{L}(T)\subset\mathrm{Hom}(\mathbb{V}\otimes\mathbb{V},\mathbb{V})$ and $g\in\mathrm{GL}(\mathbb{V})$. Thus, $\mathbb{L}(T)$ is decomposed into $\mathrm{GL}(\mathbb{V})$-orbits that correspond to the isomorphism classes of algebras. Let $\mathcal{O}(\mu)$ denote the orbits of $\mu\in\mathbb{L}(T)$ under the action of $\mathrm{GL}(\mathbb{V})$ and $\overline{\mathcal{O}(\mu)}$ denote the Zariski closure of $\mathcal{O}(\mu)$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two n-dimensional algebras satisfying the identities from T, and let $\mu,\lambda\in\mathbb{L}(T)$ represent $\mathbf{A}$ and $\mathbf{B}$, respectively. We say that $\mathbf{A}$ degenerates to $\mathbf{B}$ and write $\mathbf{A}\to\mathbf{B}$ if $\lambda\in\overline{\mathcal{O}(\mu)}$. Note that in this case we have $\overline{\mathcal{O}(\lambda)}\subset\overline{\mathcal{O}(\mu)}$. Hence, the definition of degeneration does not depend on the choice of μ and λ.

If $\mathbf{A}\not\cong\mathbf{B}$, then the assertion $\mathbf{A}\to\mathbf{B}$ is called a proper degeneration. We write $\mathbf{A}\not\to\mathbf{B}$ if $\lambda\notin\overline{\mathcal{O}(\mu)}$.

Let $\mathbf{A}$ be represented by $\mu\in\mathbb{L}(T)$. Then $\mathbf{A}$ is rigid in $\mathbb{L}(T)$ if $\mathcal{O}(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an irreducible component. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra $\mathbf{A}$ is rigid in $\mathbb{L}(T)$ if and only if $\overline{\mathcal{O}(\mu)}$ is an irreducible component of $\mathbb{L}(T)$.

3.2 Method of the description of degenerations of algebras

In the present work we use the methods applied to Lie algebras in [5]. First of all, if $\mathbf{A}\to\mathbf{B}$ and $\mathbf{A}\not\cong\mathbf{B}$, then $\mathfrak{Der}(\mathbf{A})<\mathfrak{Der}(\mathbf{B})$, where $\mathfrak{Der}(\mathbf{A})$ is the algebra of derivations of $\mathbf{A}$. We compute the dimensions of algebras of derivations and check the assertion $\mathbf{A}\to\mathbf{B}$ only for such $\mathbf{A}$ and $\mathbf{B}$ that $\mathfrak{Der}(\mathbf{A})<\mathfrak{Der}(\mathbf{B})$.

To prove degenerations, we construct families of matrices parameterized by t. Namely, let $\mathbf{A}$ and $\mathbf{B}$ be two algebras represented by the structures μ and λ from $\mathbb{L}(T)$ respectively. Let $e_1,\dots,e_n$ be a basis of $\mathbb{V}$ and c_{ij}^k $(1\le i,j,k\le n)$ be the structure constants of λ in this basis. If there exist $a_i^j(t)\in\mathbb{C}$ $(1\le i,j\le n,\ t\in\mathbb{C}^*)$ such that $E_i^t=\sum_{j=1}^n a_i^j(t)e_j$ $(1\le i\le n)$ form a basis of $\mathbb{V}$ for any $t\in\mathbb{C}^*$, and the structure constants of μ in the basis $E_1^t,\dots,E_n^t$ are such rational functions $c_{ij}^k(t)\in\mathbb{C}[t]$ that $c_{ij}^k(0)=c_{ij}^k$, then $\mathbf{A}\to\mathbf{B}$. In this case $E_1^t,\dots,E_n^t$ is called a parameterized basis for $\mathbf{A}\to\mathbf{B}$. In case of $E_1^t,E_2^t,\dots,E_n^t$ is a parametric basis for $\mathbf{A}\to\mathbf{B}$, it will be denoted by $\mathbf{A}\xrightarrow{(E_1^t,E_2^t,\dots,E_n^t)}\mathbf{B}$. To simplify our equations, we will use the notation $A_i=\langle e_i,\dots,e_n\rangle$ $(i=1,\dots,n)$ and write simply $A_pA_q\subset A_r$ instead of $c_{ij}^k=0$ $(i\ge p,\ j\ge q,\ k<r)$.

Let $\mathbf{A}(*):=\{\mathbf{A}(\alpha)\}_{\alpha\in I}$ be a series of algebras, and let $\mathbf{B}$ be another algebra. Suppose that for $\alpha\in I$, $\mathbf{A}(\alpha)$ is represented by the structure $\mu(\alpha)\in\mathbb{L}(T)$ and $\mathbf{B}$ is represented by the structure $\lambda\in\mathbb{L}(T)$. Then we say that $\mathbf{A}(*)\to\mathbf{B}$ if $\lambda\in\overline{\{\mathcal{O}(\mu(\alpha))\}_{\alpha\in I}}$, and $\mathbf{A}(*)\not\to\mathbf{B}$ if $\lambda\notin\overline{\{\mathcal{O}(\mu(\alpha))\}_{\alpha\in I}}$.

Let $\mathbf{A}(*)$, $\mathbf{B}$, $\mu(\alpha)$ $(\alpha\in I)$ and λ be as above. To prove $\mathbf{A}(*)\to\mathbf{B}$ it is enough to construct a family of pairs $(f(t),g(t))$ parameterized by $t\in\mathbb{C}^*$, where $f(t)\in I$ and $g(t)\in\mathrm{GL}(\mathbb{V})$. Namely, let $e_1,\dots,e_n$ be a basis of $\mathbb{V}$ and c_{ij}^k $(1\le i,j,k\le n)$ be the structure constants of λ in this basis. If we construct $a_i^j:\mathbb{C}^*\to\mathbb{C}$ $(1\le i,j\le n)$ and $f:\mathbb{C}^*\to I$ such that $E_i^t=\sum_{j=1}^n a_i^j(t)e_j$ $(1\le i\le n)$ form a basis of $\mathbb{V}$ for any $t\in\mathbb{C}^*$, and the structure constants of $\mu(f(t))$ in the basis $E_1^t,\dots,E_n^t$ are such rational functions

$c_{ij}^k(t)\in\mathbb{C}[t]$ that $c_{ij}^k(0)=c_{ij}^k$, then $\mathbf{A}(*)\to\mathbf{B}$. In this case $E_1^t,\dots,E_n^t$ and $f(t)$ are called a parameterized basis and a parameterized index for $\mathbf{A}(*)\to\mathbf{B}$, respectively.

We now explain how to prove $\mathbf{A}(*)\not\to\mathbf{B}$. Note that if $\mathfrak{Der}(\mathbf{A}(\alpha))>\mathfrak{Der}(\mathbf{B})$ for all $\alpha\in I$ then $\mathbf{A}(*)\not\to\mathbf{B}$. One can also use the following lemma, whose proof is the same as the proof of [5, Lemma 1.5].

Lemma 3.1. *Let $\mathfrak{B}$ be a Borel subgroup of $\mathrm{GL}(\mathbb{V})$ and $\mathcal{R}\subset\mathbb{L}(T)$ be a $\mathfrak{B}$-stable closed subset. If $\mathbf{A}(*)\to\mathbf{B}$ and for any $\alpha\in I$ the algebra $\mathbf{A}(\alpha)$ can be represented by a structure $\mu(\alpha)\in\mathcal{R}$, then there is $\lambda\in\mathcal{R}$ representing $\mathbf{B}$.*

3.3 Degeneration of 3-dimensional nilalgebras

Theorem 3.1. *The graph of degenerations of algebras from the variety of 3-dimensional nilalgebras is presented below (the number at the left-hand of the figure means the dimension of the geometric variety of the algebra from the same level). In particular, the variety of 3-dimensional nilalgebras has dimension 9, two rigid algebra $\mathcal{N}_5$ and $\mathbf{N_2}$, and three irreducible components described below*

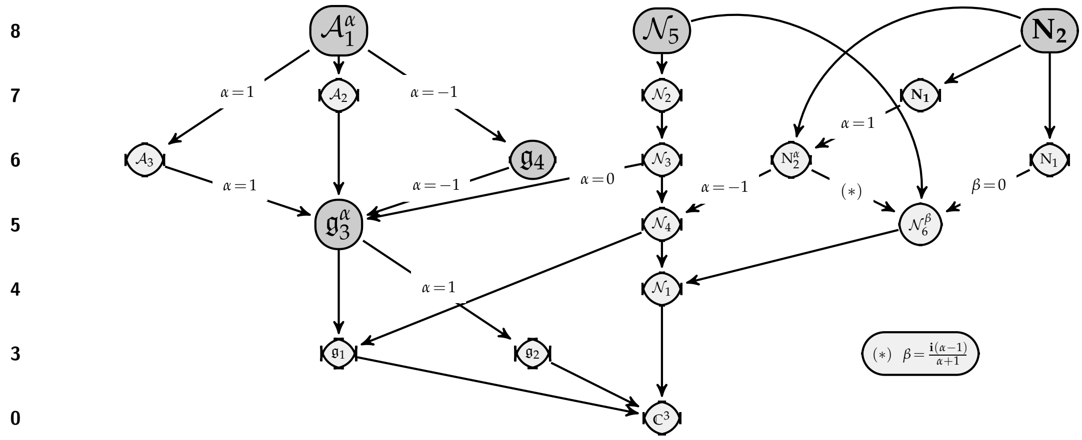
$$\begin{aligned}\overline{\mathcal{O}(\mathcal{A}_1^\alpha)}&=\{\mathcal{A}_1^\alpha,\,\mathcal{A}_2,\,\mathcal{A}_3,\,\mathfrak{g}_1,\,\mathfrak{g}_2,\,\mathfrak{g}_3^\alpha,\,\mathfrak{g}_4,\,\mathbb{C}^3\},\\ \overline{\mathcal{O}(\mathcal{N}_5)}&=\{\mathcal{N}_1,\,\mathcal{N}_2,\,\mathcal{N}_3,\,\mathcal{N}_4,\,\mathcal{N}_5,\,\mathcal{N}_6^\alpha,\,\mathfrak{g}_1,\,\mathfrak{g}_3^0,\,\mathbb{C}^3\},\\ \overline{\mathcal{O}(\mathbf{N}_2)}&=\{\mathbf{N}_1,\,\mathrm{N}_1,\,\mathrm{N}_2^\alpha,\,\mathcal{N}_1,\,\mathcal{N}_4,\,\mathcal{N}_6^\alpha,\,\mathfrak{g}_1,\,\mathbb{C}^3\}.\end{aligned}$$

In particular, the variety of 3-dimensional nilalgebras with nilindex 3 has dimension 9, one rigid algebra $\mathcal{N}_5$, and two irreducible components defined by $\mathcal{A}_1^\alpha$ and $\mathcal{N}_5$; the variety of 3-dimensional nilalgebras with nilindex 4 has dimension 9, one rigid algebra $\mathcal{N}_5$, and three irreducible components defined by $\mathcal{A}_1^\alpha$, N_2^α, and $\mathcal{N}_5$.

Proof. The subgraph of degenerations between anticommutative algebras, i.e., algebras $\mathcal{A}_i$ and $\mathfrak{g}_i$, is given in [7]. The subgraph of degenerations between nilpotent algebras, i.e., algebras $\mathfrak{g}_1$, $\mathcal{N}_1$, $\mathcal{N}_4$, $\mathcal{N}_6^\beta$, N_1, N_2^α, $\mathbf{N_1}$, and $\mathbf{N_2}$, is given in [6]. We aim to complete these subgraphs to the full graph of degenerations of 3-dimensional nilalgebras. The list of primary degenerations is given below

$$\mathcal{N}_5\xrightarrow{(\frac{1}{t}e_3,\frac{1}{t^2}e_2+te_3,-e_1)}\mathcal{N}_2,\quad \mathcal{N}_5\xrightarrow{(2t(\beta+t)e_1+2t^2(\beta+t)e_3,4t^2(\beta+t)e_3,-2t(\beta+t)^2e_2+2t(\beta+t)e_3)}\mathcal{N}_6^\beta,$$
$$\mathcal{N}_2\xrightarrow{(e_1-e_3,e_3,te_3)}\mathcal{N}_3,\quad \mathcal{N}_3\xrightarrow{(te_1,t^2e_3,-te_2+te_3)}\mathcal{N}_4,\quad \mathcal{N}_3\xrightarrow{(-\frac{1}{t}e_2+e_3,\frac{1}{t}e_2,-te_1)}\mathfrak{g}_3^0.$$

The list of primary non-degenerations is given below



Non-degenerations reasons	
$\mathcal{N}_2 \not\to \mathcal{N}_6^{\beta}$	$\mathcal{R}=\{A_2^2=0,\ c_{12}^3+c_{21}^3=c_{13}^2+c_{31}^2=0\}$
$\mathcal{N}_5 \not\to \mathfrak{g}_2,\ \mathfrak{g}_3^{\alpha\neq 0}$	$\mathcal{R}=\{A_1^2\subseteq A_2,\ c_{12}^2=c_{21}^2,\ c_{13}^2=c_{31}^2,\ 2c_{12}^2+c_{13}^3+c_{31}^3=0\ \}$

The proof is complete. □

Let us remember that the variety of 3-dimensional nilpotent algebras is irreducible and defined by a rigid algebra, the variety of n-dimensional ($n>3$) nilpotent algebras is irreducible but does not have rigid algebras [9]. The present observation and Theorem 3.1 gives the following question.
Open question. Are there rigid algebras in the variety of n-dimensional ($n>3$) nilalgebras with nilindex k?

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