

Continuous Spectrum for a Class of Smooth Mixing CMV Matrices

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Abstract. This note proves that the extended CMV matrices with Verblunsky coefficient that is generated by a smooth volume preserving mixing dynamical system and a Hölder sampling function have almost surely continuous spectrum.

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1 Introduction

Let us consider the extended CMV matrix, which is a special five-diagonal doubly infinite matrix in the standard basis of $\ell^2(\mathbb{Z})$ written as

$$\mathcal{E} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\bar{\alpha}_0\alpha_{-1} & \bar{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \cdots \\ \cdots & -\rho_0\alpha_{-1} & -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ \cdots & 0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ \cdots & 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \cdots \\ \cdots & 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad (1.1)$$

where $\{\alpha_n\}_{n \in \mathbb{Z}}$ are Verblunsky coefficients belonging to $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\rho_n = \sqrt{1 - |\alpha_n|^2}$, for $n \in \mathbb{Z}$. CMV matrix was firstly formulated by Cantero *et al.* [1].

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It is a well-known fact that the extended CMV matrices can be thought of as unitary analogs of the Jacobi matrices. One expects to get the CMV analog of the spectral results of Jacobi operators, especially the discrete Schrödinger operators. The second part of Simon's monograph [7], is written in this spirit.

Gordon lemma refers to a type of result in spectral theory that proves that strong local repetition in the operator structure causes the operator to have no eigenvalues. This result about Schrödinger operators is introduced by Gordon [5]. Recently, Fillman [4] demonstrated a version of the Gordon lemma that is valid for CMV matrices, and he applied it to show that CMV matrices with Sturmian Verblunsky coefficients have purely singular continuous spectrum supported on a Cantor set of zero Lebesgue measure for all irrational frequencies and all phases.

In 2019, Fayad and Qu [3] constructed a smooth reparametrization of a linear flow on $\mathbb{T}^3$, combining mixing with the existence of super-recurrence times for almost every point (see the relevant definitions below), and then proved that the strong recurrence implies a Gordon property on the potential, which leads to the absence of a pure point spectrum for the corresponding Schrödinger operator.

This note is devoted to get the CMV analog of the results in [3] by using the Gordon lemma for CMV matrices. Our main result and the proof will be given in the next section, i.e., Theorem 2.1.

Before state our main theorem, we would like to start with some definitions and notations.

Definition 1.1 ([4]). $\alpha \in \mathbb{D}^{\mathbb{Z}}$ is called a Gordon sequence if it is bounded away from $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ and there exist positive integers $n_k \rightarrow \infty$ such that

$$\lim_{k \rightarrow \infty} C^{n_k} \max_{0 \leq j \leq n_k - 1} |\alpha(j) - \alpha(j \pm n_k)| = 0$$

for all $C > 0$.

Definition 1.2 ([3]). Let (Ω, T) be a dynamical system with Ω a compact metric space and T a homeomorphism, i.e., topological dynamical system. Assume that $y \in \Omega$. If there exist $\beta > 1$ and an integer sequence $k_n \uparrow \infty$ such that

$$d(T^{k_n} y, y) \leq \exp(-k_n^\beta),$$

then we say that y is super-recurrent with recurrent exponent β .

If μ is an invariant ergodic measure of T , we say that the system (Ω, T, μ) is super-recurrent if μ -almost every $x \in \Omega$ is super-recurrent.

The translation of vector $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ on the n torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is the transformation

$$\mathbb{T}^n \rightarrow \mathbb{T}^n,$$