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Continuous Spectrum for a Class of Smooth Mixing CMV Matrices

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Abstract. This note proves that the extended CMV matrices with Verblunsky coefficient that is generated by a smooth volume preserving mixing dynamical system and a Hölder sampling function have almost surely continuous spectrum.

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1 Introduction

Let us consider the extended CMV matrix, which is a special five-diagonal doubly infinite matrix in the standard basis of $\ell^2(\mathbb{Z})$ written as

$$\mathcal{E}=\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\overline{\alpha}_0\alpha_{-1} & \overline{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \cdots \\ \cdots & -\rho_0\alpha_{-1} & -\overline{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ \cdots & 0 & \overline{\alpha}_2\rho_1 & -\overline{\alpha}_2\alpha_1 & \overline{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ \cdots & 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \cdots \\ \cdots & 0 & 0 & 0 & \overline{\alpha}_4\rho_3 & -\overline{\alpha}_4\alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{1.1}$$

where $\{\alpha_n\}_{n\in\mathbb{Z}}$ are Verlunsky coefficients belonging to $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ and $\rho_n=\sqrt{1-|\alpha_n|^2}$, for $n\in\mathbb{Z}$. CMV matrix was firstly formulated by Cantero *et al.* [1].

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It is a well-known fact that the extended CMV matrices can be thought of as unitary analogs of the Jacobi matrices. One expects to get the CMV analog of the spectral results of Jacobi operators, especially the discrete Schrödinger operators. The second part of Simon's monograph [7], is written in this spirit.

Gordon lemma refers to a type of result in spectral theory that proves that strong local repetition in the operator structure causes the operator to have no eigenvalues. This result about Schrödinger operators is introduced by Gordon [5]. Recently, Fillman [4] demonstrated a version of the Gordon lemma that is valid for CMV matrices, and he applied it to show that CMV matrices with Sturmian Verblunsky coefficients have purely singular continuous spectrum supported on a Cantor set of zero Lebesgue measure for all irrational frequencies and all phases.

In 2019, Fayad and Qu [3] constructed a smooth reparametrization of a linear flow on $\mathbb{T}^3$, combining mixing with the existence of super-recurrence times for almost every point (see the relevant definitions below), and then proved that the strong recurrence implies a Gordon property on the potential, which leads to the absence of a pure point spectrum for the corresponding Schrödinger operator.

This note is devoted to get the CMV analog of the results in [3] by using the Gordon lemma for CMV matrices. Our main result and the proof will be given in the next section, i.e., Theorem 2.1.

Before state our main theorem, we would like to start with some definitions and notations.

Definition 1.1 ([4]). *$\alpha\in\mathbb{D}^{\mathbb{Z}}$ is called a Gordon sequence if it is bounded away from $\partial\mathbb{D}=\{z\in\mathbb{C}:|z|=1\}$ and there exist positive integers $n_k\to\infty$ such that*

$$\lim_{k\to\infty}C^{n_k}\max_{0\leq j\leq n_k-1}|\alpha(j)-\alpha(j\pm n_k)|=0$$

for all $C>0$.

Definition 1.2 ([3]). *Let (Ω,T) be a dynamical system with Ω a compact metric space and T a homeomorphism, i.e., topological dynamical system. Assume that $y\in\Omega$. If there exist $\beta>1$ and an integer sequence $k_n\uparrow\infty$ such that*

$$d(T^{k_n}y,y)\leq\exp\left(-k_n^{\beta}\right),$$

then we say that y is super-recurrent with recurrent exponent β.

If μ is an invariant ergodic measure of T, we say that the system (Ω,T,μ) is super-recurrent if μ-almost every $x\in\Omega$ is super-recurrent.

The translation of vector $\boldsymbol{\eta}=(\eta_1,\dots,\eta_n)\in\mathbb{R}^n$ on the n torus $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ is the transformation

$$\mathbb{T}^n\to\mathbb{T}^n,$$

$$(y_1,\ldots,y_n) \to (y_1+\eta_1,\ldots,y_n+\eta_n),$$

and we denote it by $R_{\boldsymbol{\eta}}$. The translation $R_{\boldsymbol{\eta}}$ is said to be irrational if the real numbers $1,\eta_1,\ldots,\eta_n$ are rationally independent.

The translation flow on $\mathbb{T}^n$ of vector $\boldsymbol{\eta} \in \mathbb{R}^n$ is the flow arising from the constant vector field $X(y) = \boldsymbol{\eta}$. Let $\{R_{t\boldsymbol{\eta}}\}$ denote this flow and it is strictly ergodic (uniquely ergodic and minimal) for the Harr measure μ if the number $\eta_1,\ldots,\eta_n$ are rationally independent. In this case we say it is an irrational flow. Given an irrational translation $R_{\boldsymbol{\eta}}$ on $\mathbb{T}^n$, then the flow $\{R_{t(1,\boldsymbol{\eta})}\}$ on $\mathbb{T}^{n+1}$ is irrational.

If ϕ is a strictly positive smooth real function on $\mathbb{T}^n$, the reparametrization of $\{R_{t\boldsymbol{\eta}}\}$ with velocity ϕ is defined as the flow given by the vector $\phi(y)\boldsymbol{\eta}$, that is, by the system

$$\frac{dy}{dt} = \phi(y)\boldsymbol{\eta}.$$

The new flow has the same orbits as $\{R_{t\boldsymbol{\eta}}\}$ and preserves a measure equivalent to the Haar measure given by the density $1/\phi$. Moreover, if $\{R_{t\boldsymbol{\eta}}\}$ is uniquely ergodic, then so is the reparametrization flow (see [6]).

Consider a function $g \in L^1(\mathbb{T}^n)$, $g > c > 0$, where the c is an absolute constant. The special flow constructed over $R_{\boldsymbol{\eta}}$ and under the ceiling function g is the quotient flow of the action

$$\begin{aligned} &\mathbb{T}^n \times \mathbb{R} \to \mathbb{T}^n \times \mathbb{R}, \\ &(y,s) \to (y,s+t) \end{aligned}$$

by the relation $(y,s+g(y)) \sim (R_{\boldsymbol{\eta}}(y),s)$. This flow acts on the manifold $M_{R_{\boldsymbol{\eta}},g} = \mathbb{T}_n \times \mathbb{R}/\sim$.

A flow $\{P_t\}$ preserving a measure ν on M is said to be mixing if, for any measurable subsets A and B of M, one has

$$\lim_{t\to\infty} \nu\big(P^t(A) \cap B\big) = \nu(A)\nu(B).$$

In this note, we assume that T is a C^1 diffeomorphism of $\mathbb{T}^3$ and consider a sequence of Verblunsky coefficients generated by a continuous sampling function $f: \mathbb{T}^3 \to \mathbb{D}$, i.e., $\alpha_x(n) = f(T^n x)$ for $n \in \mathbb{Z}$, writing $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}_{T,\alpha,x}$.

2 Main theorem and its proof

Next lemma is a key tool in this note.

Lemma 2.1 ([4, Theorem 1.5]). *If $\alpha=\{\alpha(n)\}_{n\in\mathbb{Z}}$ is a Gordon sequence, then $\boldsymbol{\mathcal{E}}_\alpha$ has purely continuous spectrum.*

By Lemma 2.1, we know that in order to show that super-recurrence implies the absence of a point part in the spectrum, it suffices to show that it yields the Gordon sequence.

Proposition 2.1. *If $x\in\mathbb{T}^3$ is super-recurrent and $f:\mathbb{T}^3\to\mathbb{D}$ is Hölder continuous, then $\alpha_x=\{\alpha_x(n)\}_{n\in\mathbb{Z}}$ is a Gordon sequence.*

Proof. Since T is C^1, we know that T is Lipschitz and let $L>1$ be the Lipschitz constant. Suppose that f is γ-Hölder with Hölder constant C_1 and β is the recurrent exponent of x. Let $\{k_n:n\geq 1\}$ be the sequence related to x. By taking a subsequence, we can assume that $k_n\geq n$. For $1\leq l\leq k_n$, we have

$$\begin{aligned}
&\left|\alpha_x(l)-\alpha_x(l\pm k_n)\right|\\
=&\left|f(T^l x)-f(T^{l\pm k_n}x)\right|\\
\leq& C_1\left\|T^l x-T^{l\pm k_n}x\right\|^\gamma\\
\leq& C_1\left[L^l\|x-T^{\pm k_n}x\|\right]^\gamma\\
\leq& C_1\left[L^{k_n}e^{-k_n^\beta}\right]^\gamma\\
=&C_1\exp(-\gamma(k_n^\beta-k_n\ln L))\leq C^{-k_n}
\end{aligned}$$

as soon as n is large enough, where $C>0$ is a constant. By the Definition 1.1, α_x is Gordon sequence. □

We now quote two lemmas on the reparametrized flow and the special flow from [3].

Lemma 2.2 ([3, Theorem 1]). *There exist $(\eta,\eta')\in\mathbb{R}^2$ and a smooth reparametrization $\phi\in C^\infty(\mathbb{T}^3,\mathbb{R}_+^*)$ of the translation flow $\{T_{t(\eta,\eta',1)}\}$ such that the resulting flow is mixing, for its unique ergodic invariant probability measure μ_ϕ, and μ-almost every $x\in\mathbb{T}^3$ is super-recurrent for its time-one map T, where μ denotes the Haar measure on the torus $\mathbb{T}^3$ and μ_ϕ denotes the measure with density $1/\phi$.*

For $(\eta,\eta')\in\mathbb{R}^2$ and smooth function $\varphi\in C^\infty(\mathbb{T}^2,\mathbb{R}_+^*)$, let $\{R^t_{\eta,\eta',\varphi}\}$ denote the special flows over minimal translations of the two torus $\mathbb{T}^2$ and under the smooth function φ.

Lemma 2.3 ([3, Theorem 2]). *There exist a vector* $(\eta,\eta')\in\mathbb{R}^2$ *and a smooth strictly positive function* φ *defined over* $\mathbb{T}^2$ *such that the special flow* $\{R^t_{\eta,\eta',\varphi}\}$ *is mixing and* μ*-almost every* $x\in M_\varphi=\{(z,s):z\in\mathbb{T}^2,s\in[0,\varphi(z)]\}$ *is super-recurrent for* $R^1_{\eta,\eta',\varphi}$.

Let T still denote the time-one map in Lemma 2.2. From this lemma we know that μ-almost every $x\in\mathbb{T}^3$ is super-recurrent for this T and Haar measure μ, so we can prove the following theorem:

Theorem 2.1. *For every Verblunsky sequence* $\alpha_x=\{\alpha_x(n)\}_{n\in\mathbb{Z}}$ *defined by Hölder continuous sampling function* $f:\mathbb{T}^3\to\mathbb{D}$ *with* $\alpha_x(n)=f(T^nx)$, $x\in\mathbb{T}^3$, *the corresponding CMV matrix* $\boldsymbol{\mathcal{E}}_{T,\alpha,x}$ *has purely continuous spectrum for* μ*-almost every* $x\in\mathbb{T}^3$.

Proof. By [2, Section 4] we know the equivalence between special flows and reparametrizations on $\mathbb{T}^3$. And then, by Lemmas 2.2 and 2.3, we can get that μ-almost every $x\in\mathbb{T}^3$ is super-recurrent for T. Hence, Theorem 2.1 follows from Proposition 2.1 and Lemma 2.1. $\square$

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