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Numerical Ergodicity of Stochastic Allen-Cahn Equation Driven by Multiplicative White Noise

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Abstract. We establish the unique ergodicity of a fully discrete scheme for monotone SPDEs with polynomial growth drift and bounded diffusion coefficients driven by multiplicative white noise. The main ingredient of our method depends on the satisfaction of a Lyapunov condition followed by a uniform moments' estimate, combined with the regularity property for the full discretization. We transform the original stochastic equation into an equivalent random equation where the discrete stochastic convolutions are uniformly controlled to derive the desired uniform moments' estimate. Applying the main result to the stochastic Allen-Cahn equation driven by multiplicative white noise indicates that this full discretization is uniquely ergodic for any interface thickness. Numerical experiments validate our theoretical results.

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Key words: Numerical invariant measure, numerical ergodicity, stochastic Allen-Cahn equation.

1 Introduction

The invariant measure and ergodicity, as a significant long-time behavior of Markov processes generated by stochastic ordinary and partial differential equations (SODEs and SPDEs, respectively), characterize the identity of temporal aver-

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age and spatial average, which has many applications in quantum mechanics, fluid dynamics, financial mathematics, and many other fields [7, 9]. As everyone knows, the explicit expression of the invariant measure for a stochastic nonlinear system is rarely available. For this reason, it motivated and fascinated a lot of investigations in recent decades for designing numerical algorithms that can inherit the ergodicity of the original system.

There have been some developments in the construction and analysis of numerical algorithms for the invariant measures and ergodic limits of dissipative SODEs, Lipschitz SPDEs, or SPDEs with super-linear growth coefficients driven by trace-class noise. See, e.g., [10, 11, 18] and references therein for numerical ergodicity of dissipative SODEs with or without Markovian switching, [2, 4–6, 8] for approximating the invariant measures of parabolic SPDEs driven by additive noise, [14,15] for the unique ergodicity of the drift-implicit Euler Galerkin (DIEG) scheme of monotone SPDEs with polynomial growth coefficients driven by multiplicative trace-class noise.

In the settings of the infinite-dimensional case, we note that most of the existing literature focuses on the numerical ergodicity of Lipschitz SPDEs driven by additive white noise or monotone SPDEs driven by trace-class noise; the case of super-linear SPDEs driven by multiplicative white noise is more subtle and challenging. Our main aim is to indicate that the widely studied DIEG scheme (see (DIEG)) applied to second-order monotone SPDEs with polynomial growth coefficients driven by nondegenerate multiplicative white noise is uniquely ergodic (see Theorem 3.1). Applying this result to the 1D stochastic Allen-Cahn equation driven by nondegenerate multiplicative white noise indicates that its DIEG scheme is uniquely ergodic for any interface thickness (see Theorem 3.2).

The paper is organized as follows. Section 2 gives the principal assumptions on the considered SPDE. In this part, we show the unique solvability and required properties of the DIEG scheme. The Lyapunov structure and regularity property of the DIEG scheme with application to the stochastic Allen-Cahn equation are explored in Section 3. The theoretical results are verified by numerical experiments in Section 4.

2 Preliminaries

This section presents the main assumptions used throughout the paper, the solvability, and the properties needed for the full discretization to be considered. We also give some preliminaries on invariant measure and ergodicity of Markov chains.