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Equivariant Cohomology and Deformation for Associative Algebras with a Derivation

Jingru Ren*

School of Humanities and Fundamental Sciences, Shenzhen Institute of Information Technology, Shenzhen 518172, P.R. China.

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Abstract. We introduce finite group action for associative algebras equipped with a derivation (that is, AssDer pairs) and equivariant cohomology for such algebraic object. Next, we discuss equivariant deformation theory and study its relation with the equivariant cohomology.

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1 Introduction

Associative algebras are classical algebraic objects to study and has many important applications in mathematics and physics. In particular, algebraic deformation theory and Hochschild cohomology theory for associative algebras are two closely related topics and have received extensive study. Similar relations were discovered for Lie algebras, Leibniz algebras and Loday-type algebras and this is an important research direction. Derivations for associative algebras, which are a generalization of differentiation for functions, have many applications. For example, in homotopy Lie theory [17], differential Galois theory [11] and Gauge theory [1]. Recently, Tang-Frégier-Sheng [16] introduced and discussed cohomology and deformation theory for Lie algebras with derivations (called LieDer pairs).

*Corresponding author. *Email address:* `jingrur124@outlook.com` (J.R. Ren)

Later, cohomology and deformation theory for many similar algebraic structures are studied, such as AssDer pairs [2], LeibDer pairs [2], DendDer pairs [14], and these have been generalized to operads with higher derivations in [19].

Deformation theory is a fundamental research tool in mathematics, dating back at least to Riemann. Then Kodaira and Spencer introduced the idea to the study of higher dimensional complex manifolds. On the algebra side, the study was initiated by Gerstenhaber [6–9], to associative algebras.

In the late 1950s, Borel began researching equivariant cohomology, a cohomology theory for topological spaces that includes group actions. Equivariant approaches have been successfully employed to various fields, including algebraic geometry, representation theory, and K-theory. Recent research has explored the relationship between equivariant cohomology and deformation theory for various types of algebra, including Leibniz algebras [12], associative algebras [13], associative dialgebras [15], dendriform algebras [4], Lie-Yamaguti algebras [10], and Lie triple systems [18].

In the current paper, we show how the above equivariant cohomology and deformation techniques apply to algebraic objects equipped with a derivation. In particular, we introduce group actions on AssDer pairs, as well as equivariant cohomology and deformation theory for them. Similar relationships between the two are investigated and generalized to our setting. We finish the paper with a Maurer-Cartan characterization of the G-AssDer pair structure (see Definition 3.1). We work over a field $\mathbb{K}$ of characteristic zero.

2 Preliminaries

First, we recall basics about AssDer pairs and their cohomology.

Definition 2.1. *Suppose $(A,\cdot)$ is an associative algebra, a linear map $d:A\to A$ is a derivation for A, if for any $a,b\in A$ we have $d(a\cdot b)=d(a)\cdot b+a\cdot d(b)$. An associative algebra with a derivation (A,d) is called an AssDer pair.*

Definition 2.2. *Suppose (A,d) is an AssDer pair, a (A,d)-left module is a pair (M,d^M) consists of an A-left module M and $d^M:M\to M$ a linear map satisfying $d^M(am)=d(a)m+ad^M(m)$ for all $a\in A,m\in M$. Similarly one has a notion of (A,d)-right module. An (A,d)- bimodule is a pair (M,d^M) such that M is an A-bimodule and $d^M:M\to M$ is a linear map such that*

$$d^M(am)=d(a)m+ad^M(m),\quad d^M(ma)=d^M(m)a+md(a)$$

for any $a\in A,m\in M$.