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Equivariant Cohomology and Deformation for Associative Algebras with a Derivation

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Abstract. We introduce finite group action for associative algebras equipped with a derivation (that is, AssDer pairs) and equivariant cohomology for such algebraic object. Next, we discuss equivariant deformation theory and study its relation with the equivariant cohomology.

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1 Introduction

Associative algebras are classical algebraic objects to study and has many important applications in mathematics and physics. In particular, algebraic deformation theory and Hochschild cohomology theory for associative algebras are two closely related topics and have received extensive study. Similar relations were discovered for Lie algebras, Leibniz algebras and Loday-type algebras and this is an important research direction. Derivations for associative algebras, which are a generalization of differentiation for functions, have many applications. For example, in homotopy Lie theory [17], differential Galois theory [11] and Gauge theory [1]. Recently, Tang-Frégier-Sheng [16] introduced and discussed cohomology and deformation theory for Lie algebras with derivations (called LieDer pairs).

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Later, cohomology and deformation theory for many similar algebraic structures are studied, such as AssDer pairs [2], LeibDer pairs [2], DendDer pairs [14], and these have been generalized to operads with higher derivations in [19].

Deformation theory is a fundamental research tool in mathematics, dating back at least to Riemann. Then Kodaira and Spencer introduced the idea to the study of higher dimensional complex manifolds. On the algebra side, the study was initiated by Gerstenhaber [6–9], to associative algebras.

In the late 1950s, Borel began researching equivariant cohomology, a cohomology theory for topological spaces that includes group actions. Equivariant approaches have been successfully employed to various fields, including algebraic geometry, representation theory, and K-theory. Recent research has explored the relationship between equivariant cohomology and deformation theory for various types of algebra, including Leibniz algebras [12], associative algebras [13], associative dialgebras [15], dendriform algebras [4], Lie-Yamaguti algebras [10], and Lie triple systems [18].

In the current paper, we show how the above equivariant cohomology and deformation techniques apply to algebraic objects equipped with a derivation. In particular, we introduce group actions on AssDer pairs, as well as equivariant cohomology and deformation theory for them. Similar relationships between the two are investigated and generalized to our setting. We finish the paper with a Maurer-Cartan characterization of the G-AssDer pair structure (see Definition 3.1). We work over a field $\mathbb{K}$ of characteristic zero.

2 Preliminaries

First, we recall basics about AssDer pairs and their cohomology.

Definition 2.1. *Suppose $(A,\cdot)$ is an associative algebra, a linear map $d: A \to A$ is a derivation for A, if for any $a,b \in A$ we have $d(a\cdot b) = d(a)\cdot b + a\cdot d(b)$. An associative algebra with a derivation (A,d) is called an AssDer pair.*

Definition 2.2. *Suppose (A,d) is an AssDer pair, a (A,d)-left module is a pair (M,d^M) consists of an A-left module M and $d^M: M \to M$ a linear map satisfying $d^M(am) = d(a)m + ad^M(m)$ for all $a \in A, m \in M$. Similarly one has a notion of (A,d)-right module. An (A,d)- bimodule is a pair (M,d^M) such that M is an A-bimodule and $d^M: M \to M$ is a linear map such that*

$$d^M(am) = d(a)m + ad^M(m), \quad d^M(ma) = d^M(m)a + md(a)$$

for any $a \in A, m \in M$.

Remark 2.1. Any AssDer pair (A,d) is an (A,d)-bimodule over itself.

Now, let (A,d) be an AssDer pair and M an (A,d)-bimodule. Define

$$C^0(A,M):=0, \quad C^1(A,M):=Hom(A,M)$$

and

$$C^n(A,M):=Hom(A^{\otimes n},M)\times Hom(A^{\otimes n-1},M) \quad \text{for} \quad n\geqslant 2,$$

where

$$Hom(-,-)=Hom_k(-,-).$$

Define

$$\partial^n: C^n(A,M) \ \rightarrow \ C^{n+1}(A,M)$$

as

$$\begin{aligned}
&\partial^1\alpha:=(\delta_{Hoch}\alpha,-\delta\alpha),\\
&\partial^n(\alpha,\beta):=\big(\delta_{Hoch}\alpha,\delta_{Hoch}\beta+(-1)^n\delta\alpha\big) \quad \text{for} \quad n\geqslant 2,
\end{aligned}$$

where

$$\delta_{Hoch}: Hom(A^{\otimes n},M) \ \rightarrow \ Hom(A^{\otimes n+1},M)$$

stands for the classical Hochschild differential for associative algebras

$$\begin{aligned}
&(\delta_{Hoch}\alpha)(x_1,\cdots,x_{n+1})\\
:=\,&x_1\alpha(x_2,\cdots,x_{n+1})+\sum_{i=1}^{n}(-1)^i\alpha(x_1,\cdots,x_ix_{i+1},\cdots,x_{n+1})\\
&+(-1)^{n+1}\alpha(x_1,\cdots,x_n)x_{n+1}(*)
\end{aligned}$$

and

$$\delta: Hom(A^{\otimes n},M) \ \rightarrow \ Hom(A^{\otimes n},M)$$

is the map

$$\delta\alpha:=\sum_{i=1}^{n}\alpha\circ(id,\cdots,d,\cdots,id)-d^M\circ\alpha.$$

We have

Lemma 2.1. *1.* ([3, Lemma 1]) $\quad \delta_{Hoch}\circ\delta=\delta\circ\delta_{Hoch}$,

2. ([3, Proposition 4]) $\quad \partial^n\circ\partial^{n+1}=0 \quad$ *for* $\quad n\geqslant 1$.

Hence, $(C^*(A,M),\partial)$ forms a cochain complex and we define the homologies of this complex to be the cohomology of the AssDer pair (A,d) with coefficients in M and denote it by $H^*(A,M)$. We use $C^*(A)$ to denote $C^*(A,A)$ when M is taken to be A itself.

As noted in [5], the graded space $\bigoplus_n Hom(A^{\otimes n},A)$ of Hochschild cochains of the associative algebra A carries a degree -1 graded Lie bracket

$$[\alpha,\beta] := \alpha\circ\beta - (-1)^{(m-1)(n-1)}\beta\circ\alpha,$$

where

$$\begin{aligned}&(\alpha\circ\beta)(x_1,\ldots,x_{m+n-1})\\ :=&\sum_{i=1}^{m}(-1)^{(i-1)(n-1)}\alpha\big(x_1,\ldots,x_{i-1},\beta(x_i,\ldots,x_{i+n-1}),\ldots,x_{m+n-1}\big)\end{aligned}$$

for $\alpha\in Hom(A^{\otimes m},A)$ and $\beta\in Hom(A^{\otimes n},A)$. One checks directly that an element $m\in Hom(A^{\otimes 2},A)$ gives an associative multiplication if and only if $[m,m]=0$ and $d\in Hom(A,A)$ is a derivation with respect to a multiplication m if and only if $[m,d]=0$.

It is further observed that

Proposition 2.1 ([3, Proposition 5]). *There is an induced degree -1 graded Lie bracket on $\bigoplus_n C^n(A)$,*

$$[-,-]_D : C^m(A)\times C^n(A) \;\to\; C^{m+n-1}(A),$$

where

$$\big[(\alpha,\bar{\alpha}),(\beta,\bar{\beta})\big]_D := \Big([\alpha,\beta],(-1)^{m+1}[\alpha,\bar{\beta}]+[\bar{\alpha},\beta]\Big).$$

Therefore, $(\bigoplus_n C^{n+1}(A),[-,-]_D)$ is a graded Lie algebra and the differential in the AssDer pair cochain complex $(C^*(A),\partial)$ has a characterization $\partial(-)=[(\cdot_A,d),-]_D$. In fact, $(\bigoplus_n C^{n+1}(A),[-,-]_D,\partial)$ is a dg Lie algebra.

3 Equivariant cohomology for AssDer pair

We introduce the notion of group action and equivariant cohomology theory for AssDer pairs in this section. First,

Definition 3.1. *Let G be a finite group and (A,d) be an AssDer pair. Define a G-action on (A,d) to be a map*

$$\varphi: G\times A \rightarrow A, \quad (g,a) \mapsto \varphi(g,a):=g\cdot a$$

such that for $g,h\in G$, $x,y\in A$ we have

1. *$e\cdot x=x$, where e is the identity element of G,*
2. *$g\cdot(h\cdot x)=(g\cdot h)\cdot x$,*
3. *$\varphi_g(-):=\varphi(g,-): A\rightarrow A$, $a\mapsto g\cdot a$ is linear,*
4. *$g\cdot(a\cdot b)=(g\cdot a)\cdot(g\cdot b)$,*
5. *$g\cdot da=d(g\cdot a)$.*

An AssDer pair together with a G-action is referred to as a G-AssDer pair.

Note that we use the same symbol '$\cdot$' to denote both the multiplication in A and the group action. It should be clear from the context of what we really mean and we will continue to do this for the entire paper.

Definition 3.2. *Let (A,d) be a G-AssDer pair, a G-bimodule over (A,d) is an (A,d)-bimodule (M,d^M) such that G acts linearly on M, the left, right A-action on M and d^M are G-equivariant, that is, $(g\cdot a)\cdot(g\cdot m)=g\cdot(a\cdot m)$, $(g\cdot m)\cdot(g\cdot a)=g\cdot(m\cdot a)$ and $g\cdot d^M(m)=d^M(g\cdot m)$ for any $a\in A$, $m\in M$.*

Remark 3.1. Any G-AssDer pair (A,d) is naturally a G-bimodule over itself.

Suppose (A,d) is a G-AssDer pair and M a G-bimodule, define

$$\begin{aligned}
C^1_G(A,M)&:=\Big\{\alpha\in C^1(A,M)=Hom(A,M):\ \alpha(g\cdot x)=g\cdot\alpha(x)\ \text{ for any }\ g\in G\Big\},\\
C^n_G(A,M)&:=Hom_G(A^{\otimes n},M)\times Hom_G(A^{\otimes n-1},M)\\
&:=\Big\{(\alpha,\beta)\in C^n(A,M):\ \alpha(g\cdot x_1,\cdots,g\cdot x_n)=g\cdot\alpha(x_1,\cdots,x_n),\\
&\qquad \beta(g\cdot x_1,\cdots,g\cdot x_{n-1})=g\cdot\beta(x_1,\cdots,x_{n-1})\ \text{ for any }\ g\in G\Big\},\quad n\geqslant 2.
\end{aligned}$$

Elements in $Hom_G(A^{\otimes n},M)$ are called equivariant.

Lemma 3.1. *1. $\partial^1(\alpha)\in C^2_G(A,M)$, if $\alpha\in C^1_G(A,M)$.*

2. *For* $n\geqslant 2$, $\partial^2(\alpha,\beta)\in C_G^{n+1}(A,M)$, *if* $(\alpha,\beta)\in C_G^n(A,M)$.

Proof.

$$\partial^1\alpha=(\delta_{Hoch}\alpha,-\delta\alpha)\in C^2(A,M)=Hom(A^{\otimes 2},M)\times Hom(A,M)$$

with

$$(\delta_{Hoch}\alpha)(x,y)=x\alpha(y)-\alpha(x)y,$$

so

$$\begin{aligned}&(\delta_{Hoch}\alpha)(g\cdot x,g\cdot y)\\ =&(g\cdot x)\alpha(g\cdot y)-\alpha(g\cdot x)(g\cdot y)\\ =&(g\cdot x)\big(g\cdot\alpha(y)\big)-\big(g\cdot\alpha(x)\big)(g\cdot y)\\ =&g\cdot\big(x\alpha(y)\big)-g\cdot\big(\alpha(x)y\big)\\ =&\big(g(\delta_{Hoch}\alpha)\big)(x,y)\end{aligned}$$

and

$$-\delta\big(\alpha(g\cdot x)\big)=-\delta\big(g\alpha(x)\big)=-g\big(\delta(\alpha(x))\big)=g\big(-\delta(\alpha(x))\big),$$

hence $\partial^1(\alpha)\in C_G^2(A,M)$. Now, consider the case $n\geqslant 2$. First of all, for any $\gamma\in Hom_G^n(A,M)$ we get $\delta_{Hoch}\gamma\in Hom(A^{\otimes n+1},M)$ from $(*)$ (on page 2) and

$$\begin{aligned}&(\delta_{Hoch}\gamma)(gx_1,\cdots,gx_{n+1})\\ :=&(gx_1)\gamma(gx_2,\cdots,gx_{n+1})+\sum_{i=1}^n(-1)^i\gamma\big(gx_1,\cdots,(gx_i)(gx_{i+1}),\cdots,gx_{n+1}\big)\\ &\quad+(-1)^{n+1}\gamma(gx_1,\cdots,gx_n)(gx_{n+1})\\ =&(gx_1)\big(g\gamma(x_2,\cdots,x_{n+1})\big)+\sum_{i=1}^n(-1)^i\gamma\big(gx_1,\cdots,g(x_ix_{i+1}),\cdots,gx_{n+1}\big)\\ &\quad+(-1)^{n+1}\big(g\gamma(x_1,\cdots,x_n)\big)(gx_{n+1})\\ =&g\big(x_1\gamma(x_2,\cdots,x_{n+1})\big)+\sum_{i=1}^n(-1)^ig\gamma(x_1,\cdots,x_ix_{i+1},\cdots,x_{n+1})\\ &\qquad\qquad+(-1)^{n+1}g\big(\gamma(x_1,\cdots,x_n)x_{n+1}\big)\\ =&g\Bigg(x_1\gamma(x_2,\cdots,x_{n+1})+\sum_{i=1}^n(-1)^i\gamma(x_1,\cdots,x_ix_{i+1},\cdots,x_{n+1})\\ &\qquad\qquad+(-1)^{n+1}\gamma(x_1,\cdots,x_n)x_{n+1}\Bigg)\\ =&\big(g(\delta_{Hoch}\gamma)\big)(x_1,\cdots,x_{n+1}).\end{aligned}$$

Similarly, we also have

$$
\begin{aligned}
&(\delta\gamma)(g\cdot x_1,\cdots,g\cdot x_n)\\
=&\sum_{i=1}^{n}\gamma\big(g\cdot x_1,\cdots,d(g\cdot x_i),\cdots,g\cdot x_n\big)-d\big(\gamma(g\cdot x_1,\cdots,g\cdot x_n)\big)\\
=&\sum_{i=1}^{n}\gamma\big(g\cdot x_1,\cdots,g\cdot d(x_i),\cdots,g\cdot x_n\big)-d\big(g\cdot\gamma(x_1,\cdots,x_n)\big)\\
=&\sum_{i=1}^{n}g\cdot\gamma\big(x_1,\cdots,d(x_i),\cdots,x_n\big)-g\cdot d\big(\gamma(x_1,\cdots,x_n)\big)\\
=&\big(g\cdot(\delta\gamma)\big)(x_1,\cdots,x_n),
\end{aligned}
$$

so that for $(\alpha,\beta)\in C^n_G(A,M)$, we have

$$
\begin{aligned}
&\big(\partial^n(\alpha,\beta)\big)(g\cdot x_1,\cdots,g\cdot x_{n+1})\\
=&\Big((\delta_{Hoch}\alpha)(g\cdot x_1,\cdots,g\cdot x_{n+1}),(\delta_{Hoch}\beta)(g\cdot x_1,\cdots,g\cdot x_{n+1})\\
&\quad+(-1)^n(\delta\alpha)(g\cdot x_1,\cdots,g\cdot x_{n+1})\Big)\\
=&\Big(\big(g\cdot(\delta_{Hoch}\alpha)\big)(x_1,\cdots,x_{n+1}),\big(g\cdot(\delta_{Hoch}\beta)\big)(x_1,\cdots,x_{n+1})\\
&\quad+(-1)^n\big(g\cdot(\delta\alpha)\big)(x_1,\cdots,x_{n+1})\Big)\\
=&\big(g\cdot(\partial(\alpha,\beta))\big)(x_1,\cdots,x_{n+1}).
\end{aligned}
$$

The proof is complete. □

Thus, $(C^*_G(A,M),\partial)$ is also a cochain complex. Therefore, we can make the following definition.

Definition 3.3. *Suppose (A,d) is a G-AssDer pair and M a G-bimodule, define the homologies of the complex $(C^*_G(A,M),\partial)$ to be the equivariant cohomology of (A,d) with coefficients in M and denote it by $H^*_G(A,M)$.*

4 Equivariant deformation

Denote by $A[[t]]$ the space of formal power series, so $A[[t]]$ is an $\mathbb{K}[[t]]$-module.

Definition 4.1. *An equivariant deformation for a G-AssDer pair (A,d) consists of formal power series*

$$m_t: A[[t]]\times A[[t]] \rightarrow A[[t]], \quad d_t: A[[t]] \rightarrow A[[t]],$$
$$m_t(a,b)=a\cdot b+\sum_{i=1}^{\infty} m_i(a,b)t^i, \quad d_t(a)=d(a)+\sum_{i=1}^{\infty} d_i(a)t^i$$

such that $(A[[t]],m_t,d_t)$ is an AssDer pair over $\mathbb{K}[[t]]$, where $m_i(ga,gb)=g\cdot m_i(a,b)$, $d_i(ga)=g\cdot d_i(a)$ (that is, $m_i\in Hom_G(A\otimes A,A)$, $d_i\in Hom_G(A,A)$).

The operation m_t satisfies associativity law exactly like classical algebraic deformation theory so for $a,b,c\in A$, we have

$$m_t\big(m_t(a,b),c\big)=m_t\big(a,m_t(b,c)\big)$$
$$\Leftrightarrow \sum_{p+q=r,p,q\geqslant 0} m_p\big(m_q(a,b),c\big)-m_p\big(a,m_q(b,c)\big)=0.$$

Similarly,

$$d_t\big(m_t(a,b)\big)=m_t\big(d_t(a),b\big)+m_t\big(a,d_t(b)\big)$$
$$\Leftrightarrow \sum_{p+q=r,p,q\geqslant 0} d_p\big(m_q(a,b)\big)=\sum_{p+q=r,p,q\geqslant 0} m_p\big(d_q(a),b\big)+m_p\big(a,d_q(b)\big).$$

In particular,

$$m_1(ab,c)+m_1(a,b)c=am_1(b,c)+m_1(a,bc),$$
$$d\big(m_1(a,b)\big)+d_1(ab)=d_1(a)b+m_1\big(d(a),b\big)+ad_1(b)+m_1\big(a,d(b)\big),$$

that is,

$$\partial^1(m_1,d_1)=\delta_{Hoch}d_1+\delta m_1=0.$$

In addition, since m_i,d_i are equivariant, we have

Proposition 4.1. *Let (m_t,d_t) be an equivariant deformation for the G-AssDer pair (A,d). Then (m_1,d_1) is a 2-cocycle for the self coefficient equivariant cohomology for (A,d).*

The 2-cocycle (m_1,d_1) is called the infinitesimal of the equivariant deformation (m_t,d_t).

Definition 4.2. *Suppose (m_t,d_t) and (m'_t,d'_t) are two equivariant deformations for a G-AssDer pair (A,d), suppose there is an equivariant formal isomorphism*

$$\varphi_t=id_A+\sum_{i=1}^{\infty}\varphi_i t^i: A[[t]] \rightarrow A[[t]],$$

that is, each φ_i is a formal isomorphism from A to itself such that

$$m'_t \circ (\varphi_t \otimes \varphi_t) = \varphi_t \circ m_t, \quad \varphi_t \circ d_t = d'_t \circ \varphi_t.$$

Then we say (m_t, d_t) and (m'_t, d'_t) are equivalent.

Equivalently, this means

$$\sum_{i+j+k=n} m'_i \circ (\varphi_j \otimes \varphi_k) = \sum_{i+j=n} \varphi_i \circ m_j, \quad \sum_{i+j=n} \varphi_i \circ d_j = \sum_{i+j=n} d'_i \circ \varphi_j,$$

so for $n=1$, we have

$$\begin{aligned} &m_1 + \varphi_1 \circ m = m'_1 + m \circ (\varphi_1 \otimes id) + m \circ (id \otimes \varphi_1), \\ &d_1 + \varphi_1 \circ d = d'_1 + d \circ \varphi_1, \end{aligned}$$

hence $(m_1, d_1) - (m'_1, d'_1) = \partial(\varphi_1)$, so

Proposition 4.2. *Two equivalent equivariant deformations have cohomologous infnitesimals, that is, the infnitesimals of two equivalent equivariant deformations are cohomologous.*

Definition 4.3. *We say (m_t, d_t) is trivial if it is equivalent to $(m_0, d_0) = (m, d)$ and the G-AssDer pair (A, d) is rigid if all equivariant deformations are trivial.*

Theorem 4.1. *(A, d) is rigid if $H^2_G(A, A) = 0$.*

Proof. Let (m_t, d_t) be an equivariant deformation for (A, d) so (m_1, d_1) is a 2-cocycle by Proposition 4.1, hence there is some $\varphi_1 \in C^1_G(A, A) = Hom_G(A, A)$ such that $(m_1, d_1) = \partial^1(\varphi_1)$. Let

$$\varphi_t := id_A + \varphi_1 t : A[[t]] \ \to \ A[[t]],$$

define

$$m'_t := \varphi_t^{-1} \circ m_t \circ (\varphi_t \otimes \varphi_t), \quad d'_t := \varphi_t^{-1} \circ d_t \circ \varphi_t,$$

so (m'_t, d'_t) is equivalent to (m_t, d_t). From above definition we know

$$m'_t = m + m'_2 t^2 + \cdots, \quad d'_t = d + d'_2 t^2 + \cdots,$$

that is, the linear part of (m'_t, d'_t) is canceled. Repeat the above process and eventually this gives (m_t, d_t) and (m, d) are equivalent. ◻

Definition 4.4. *Let (A,d) be a G-AssDer pair, an order n equivariant deformation consists of formal power series*

$$m_t(a,b)=a\cdot b+\sum_{i=1}^{n}m_i(a,b)t^i,\quad d_t(a)=d(a)+\sum_{i=1}^{n}d_i(a)t^i$$

such that $(A[[t]]/(t^{n+1}),m_t,d_t)$ is an AssDer pair, with $m_i\in Hom_G(A\otimes A,A)$, $d_i\in Hom_G(A,A)$.

Explicitly, for any order n equivariant deformation, we have

$$\sum_{i+j=k}m_i\big(m_j(a,b),c\big)=\sum_{i+j=k}m_i\big(a,m_j(b,c)\big),\tag{4.1}$$

$$\sum_{i+j=k}d_i\big(m_j(a,b)\big)=\sum_{i+j=k}m_i\big(d_j(a),b\big)+m_i\big(a,d_j(b)\big)\tag{4.2}$$

for $k=0,1,\ldots,n$.

Definition 4.5. *Let (m_t,d_t) be an order n equivariant deformation, if there is some $(m_{n+1},d_{n+1})\in C^2_G(A,A)$ such that*

$$\Big(m'_t:=m_t+m_{n+1}t^{n+1},\,d'_t:=d_t+d_{n+1}t^{n+1}\Big)$$

is an order $n+1$ equivariant deformation. Then we say the equivariant deformation (m_t,d_t) is extensible.

Hence, for any extensible order n equivariant deformation (m_t,d_t), the following two formulas need to hold in addition to (4.1) and (4.2):

$$\sum_{i+j=n+1}m_i\big(m_j(a,b),c\big)=\sum_{i+j=n+1}m_i\big(a,m_j(b,c)\big),$$
$$\sum_{i+j=n+1}d_i\big(m_j(a,b)\big)=\sum_{i+j=n+1}m_i\big(d_j(a),b\big)+m_i\big(a,d_j(b)\big).$$

Equivalently,

$$\begin{aligned}&\delta_{Hoch}(m_{n+1})(a,b,c)\\=&\sum_{i+j=n+1,i,j>0}m_i\big(m_j(a,b),c\big)-m_i\big(a,m_j(b,c)\big)\\=:&Ob^3(a,b,c),\\&(\delta_{Hoch}d_{n+1}+\delta m_{n+1})(a,b)\\=&\sum_{i+j=n+1,i,j>0}d_i\big(m_j(a,b)\big)-m_i\big(d_j(a),b\big)-m_i\big(a,d_j(b)\big)\\=:&Ob^2(a,b).\end{aligned}$$

Proposition 4.3. *$Ob:=(Ob^3,Ob^2)$ is a 3-cocycle. This is called the obstruction class of (m_t,d_t).*

Proof. (Ob^3,Ob^2) is a 3-cocycle for an ordinary AssDer pair (that is, without a finite group action) [3] so we just need to check whether everything still holds for the equivariant situation and this is obvious, as all the m_i, d_i are themselves equivariant. □

Theorem 4.2. *Any order n equivariant deformation (m_t,d_t) is extensible if and only if the obstruction class is trivial.*

Corollary 4.1. *Every finite order equivariant deformation is extensible if $H^3_G(A,A)=0$.*

Corollary 4.2. *Every 2-cocycle of the equivariant cohomology is the infinitesimal of some equivariant deformation if $H^3_G(A,A)=0$.*

5 Maurer-Cartan characterization

We finish the paper with a Maurer-Cartan characterization of G-AssDer pairs. Let (A,d) be a G-AssDer pair throughout the section. We begin with the following lemma.

Lemma 5.1. *For any $\alpha\in Hom_G(A^{\otimes m},A)$ and $\beta\in Hom_G(A^{\otimes n},A)$, we have $\alpha\circ\beta\in Hom_G(A^{\otimes m+n-1},A)$. Thus, $[(\alpha,\bar\alpha),(\beta,\bar\beta)]_D\in C^{m+n-1}_G(A)$ if $(\alpha,\bar\alpha)\in C^m_G(A)$ and $(\beta,\bar\beta)\in C^n_G(A)$.*

Proof. Indeed,

$$
\begin{aligned}
&(\alpha\circ\beta)(g\cdot x_1,\dots,g\cdot x_{m+n-1})\\
=&\sum_{i=1}^m(-1)^{(i-1)(n-1)}\alpha\big(g\cdot x_1,\dots,g\cdot x_{i-1},\beta(g\cdot x_i,\dots,g\cdot x_{i+n-1}),\dots,g\cdot x_{m+n-1}\big)\\
=&\sum_{i=1}^m(-1)^{(i-1)(n-1)}\alpha\big(g\cdot x_1,\dots,g\cdot x_{i-1},g\cdot\beta(x_i,\dots,x_{i+n-1}),\dots,g\cdot x_{m+n-1}\big)\\
=&\sum_{i=1}^m(-1)^{(i-1)(n-1)}g\cdot\alpha\big(x_1,\dots,x_{i-1},\beta(x_i,\dots,x_{i+n-1}),\dots,x_{m+n-1}\big)\\
=&\big(g\cdot(\alpha\circ\beta)\big)(x_1,\dots,x_{m+n-1}).
\end{aligned}
$$

The proof is complete. □

Therefore, the above lemma and Proposition 2.1 implies

Proposition 5.1. *The bracket* $[-,-]_D$ *on* $\bigoplus_n C^n(A)$ *gives an induced degree* -1 *graded Lie bracket on* $\bigoplus_n C^n_G(A)$*. Therefore,* $(\bigoplus_n C^{n+1}_G(A),[-,-]_D)$ *is a graded Lie algebra.*

Corollary 5.1. *A pair* $(m,d)\in C^2_G(A)$ *defines a G-AssDer pair on* A *if and only if it is a Maurer-Cartan element of the graded Lie algebra* $(\bigoplus_n C^{n+1}_G(A),[-,-]_D)$.

Proof. An element $(m,d)\in C^2_G(A)$ is a Maurer-Cartan element of the graded Lie algebra $(\bigoplus_n C^{n+1}_G(A),[-,-]_D)$ if

$$0=[(m,d),(m,d)]_D=([m,m],2[m,d]).$$

Hence, (A,m,d) is a G-AssDer pair if and only if it is a Maurer-Cartan element. □

By above discussion and observation at the end of Section 2, we have

Corollary 5.2. $(\bigoplus_n C^{n+1}_G(A),[-,-]_D,\partial(-)=[(\cdot_A,d),-]_D)$ *is a dg Lie algebra for any G-AssDer pair* $(A,\cdot_A,d)$.

We finish with the following further characterization of the G-AssDer structure. Note that we use m,m' to denote the associative multiplication $\cdot_A$ and $\cdot'_A$ below to avoid weird notation like $\cdot_A+\cdot'_A$.

Theorem 5.1. *Let* (A,m,d) *be a G-AssDer pair and* $(m',d')\in C^2_G(A)$*. Then* $(A,m+m',d+d')$ *is a G-AssDer pair if and only if* (m',d') *is a Maurer-Cartan element of the dg Lie algebra*

$$\left(\bigoplus_n C^{n+1}_G(A),[-,-]_D,\partial(-)=[(\cdot_A,d),-]_D\right),$$

i.e., (m',d') *satisfies the Maurer-Cartan equation*

$$\partial(m',d')+\frac{1}{2}\big[(m',d'),(m',d')\big]_D=0.$$

Proof. By Corollary 5.1, $(A,m+m',d+d')$ is a G-AssDer pair if and only if $[(m+m',d+d'),(m+m',d+d')]_D=0$. Note

$$\begin{aligned}&\big[(m+m',d+d'),(m+m',d+d')\big]_D\\ &=2\big[(m,d),(m',d')\big]_D+\big[(m',d'),(m',d')\big]_D\\ &=2\left(\partial(m',d')+\frac{1}{2}\big[(m',d'),(m',d')\big]_D\right),\end{aligned}$$

because $[(m,d),(m,d)]_D=0$. □

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