

Commutators of Complex Symmetric Operators

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Abstract. Let C be a conjugation on a separable complex Hilbert space $\mathcal{H}$. An operator T on $\mathcal{H}$ is said to be C -symmetric if $CTC = T^*$, and T is said to be C -skew symmetric if $CTC = -T^*$. It is proved in this paper that each C -skew symmetric operator can be written as the sum of two commutators of C -symmetric operators.

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1 Introduction

This paper is a continuation of [28], where some results were obtained to exhibit connections between complex symmetric operators and skew symmetric operators. The aim of the present paper is to represent skew symmetric operators in terms of complex symmetric operators. To proceed, we first introduce some notations and terminology.

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Throughout this paper, we denote by $\mathcal{H}$ a separable complex Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$, and by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A map C on $\mathcal{H}$ is called a conjugation, if C is conjugate-linear, invertible with $C^{-1} = C$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$.

Definition 1.1. (i) An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be complex symmetric if $CTC = T^*$ for some conjugation C on $\mathcal{H}$; in this concrete case, T is also called a C -symmetric operator.

(ii) An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be skew symmetric if $CTC = -T^*$ for some conjugation C on $\mathcal{H}$; in this concrete case, T is also called a C -skew symmetric operator.

Complex symmetric operators are natural generalizations of symmetric matrices and have been studied in the finite dimensional case for many years. The general study of complex symmetric operators was initiated by Garcia, Putinar and Wogen [16–18], and has received much attention in the past nearly two decades. Many significant results have been obtained and show that complex symmetric operators are closely related to the study of concrete operators [12–14, 26] as well as mathematics physics [15, 20, 25]. In particular, it is worth mentioning that the study of complex symmetric operators is closely related to that of truncated Toeplitz operators, initiated in Sarason's seminal paper [26]. The reader is referred to [4, 6, 19, 22–24, 30] for more results on complex symmetric operators.

The study of skew symmetric operators has classical roots in algebra and geometry. Skew symmetric operators are natural generalizations of skew symmetric matrices. In fact, an operator $T \in \mathcal{B}(\mathcal{H})$ is skew symmetric if and only if T can be written as a skew symmetric matrix with respect to some orthonormal basis of $\mathcal{H}$. The Lie algebra consisting of $n \times n$ skew symmetric matrices is one of the classical finite-dimensional Lie algebras. Its infinite-dimensional analogue $\mathcal{O}_C$, called the orthogonal Lie algebra of operators, is consisting of all C -skew symmetric operators for some conjugation C on some infinite-dimensional complex Hilbert space $\mathcal{H}$. In [11], de La Harpe discussed in detail the ideals, derivations, real forms and automorphisms of $\mathcal{O}_C$. In a recent paper [7], Bu and the third author determined the Lie ideals of $\mathcal{O}_C$, their dual spaces as well as the spectra of the derivations on $\mathcal{O}_C$. The reader is referred to [2, 3, 8, 21, 27, 29] for more results about the operator-theoretic aspects of skew symmetric operators.

Complex symmetric operators and skew symmetric operators are closely connected to each other. For C a conjugation, the set $\mathcal{S}_C$ of all C -symmetric operators is called the Hermitian type Cartan factor, while $\mathcal{O}_C$ is called the symplectic type Cartan factor [10]. They both appear in the study of JB^* -triples and play an important role in classification of bounded symmetric domains (see [9, 10]). One