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Commutators of Complex Symmetric Operators

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Abstract. Let C be a conjugation on a separable complex Hilbert space $\mathcal{H}$. An operator T on $\mathcal{H}$ is said to be C-symmetric if $CTC=T^*$, and T is said to be C-skew symmetric if $CTC=-T^*$. It is proved in this paper that each C-skew symmetric operator can be written as the sum of two commutators of C-symmetric operators.

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1 Introduction

This paper is a continuation of [28], where some results were obtained to exhibit connections between complex symmetric operators and skew symmetric operators. The aim of the present paper is to represent skew symmetric operators in terms of complex symmetric operators. To proceed, we first introduce some notations and terminology.

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Throughout this paper, we denote by $\mathcal{H}$ a separable complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, and by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A map C on $\mathcal{H}$ is called a conjugation, if C is conjugate-linear, invertible with $C^{-1}=C$ and $\langle Cx,Cy\rangle=\langle y,x\rangle$ for all $x,y\in\mathcal{H}$.

Definition 1.1. *(i) An operator $T\in\mathcal{B}(\mathcal{H})$ is said to be complex symmetric if $CTC=T^*$ for some conjugation C on $\mathcal{H}$; in this concrete case, T is also called a C-symmetric operator.*

(ii) An operator $T\in\mathcal{B}(\mathcal{H})$ is said to be skew symmetric if $CTC=-T^*$ for some conjugation C on $\mathcal{H}$; in this concrete case, T is also called a C-skew symmetric operator.

Complex symmetric operators are natural generalizations of symmetric matrices and have been studied in the finite dimensional case for many years. The general study of complex symmetric operators was initiated by Garcia, Putinar and Wogen [16–18], and has received much attention in the past nearly two decades. Many significant results have been obtained and show that complex symmetric operators are closely related to the study of concrete operators [12–14, 26] as well as mathematics physics [15, 20, 25]. In particular, it is worth mentioning that the study of complex symmetric operators is closely related to that of truncated Toeplitz operators, initiated in Sarason's seminal paper [26]. The reader is referred to [4, 6, 19, 22–24, 30] for more results on complex symmetric operators.

The study of skew symmetric operators has classical roots in algebra and geometry. Skew symmetric operators are natural generalizations of skew symmetric matrices. In fact, an operator $T\in\mathcal{B}(\mathcal{H})$ is skew symmetric if and only if T can be written as a skew symmetric matrix with respect to some orthonormal basis of $\mathcal{H}$. The Lie algebra consisting of $n\times n$ skew symmetric matrices is one of the classical finite-dimensional Lie algebras. Its infinite-dimensional analogue $\mathcal{O}_C$, called the orthogonal Lie algebra of operators, is consisting of all C-skew symmetric operators for some conjugation C on some infinite-dimensional complex Hilbert space $\mathcal{H}$. In [11], de La Harpe discussed in detail the ideals, derivations, real forms and automorphisms of $\mathcal{O}_C$. In a recent paper [7], Bu and the third author determined the Lie ideals of $\mathcal{O}_C$, their dual spaces as well as the spectra of the derivations on $\mathcal{O}_C$. The reader is referred to [2, 3, 8, 21, 27, 29] for more results about the operator-theoretic aspects of skew symmetric operators.

Complex symmetric operators and skew symmetric operators are closely connected to each other. For C a conjugation, the set $\mathcal{S}_C$ of all C-symmetric operators is called the Hermitian type Cartan factor, while $\mathcal{O}_C$ is called the symplectic type Cartan factor [10]. They both appear in the study of JB*-triples and play an important role in classification of bounded symmetric domains (see [9, 10]). One can check that $\mathcal{S}_C$ and $\mathcal{O}_C$ are complementary closed subspaces of $\mathcal{B}(\mathcal{H})$, that is, $\mathcal{S}_C+\mathcal{O}_C=\mathcal{B}(\mathcal{H})$ and $\mathcal{S}_C\cap\mathcal{O}_C=\{0\}$. In [28], more connections between $\mathcal{S}_C$ and $\mathcal{O}_C$ were established. It was proved that $\mathcal{S}_C$ and $\mathcal{O}_C$ are Roberts orthogonal to each other (see [28, Theorem 2.1]), that is, $\|A-\lambda B\|=\|A+\lambda B\|$ for all $A\in\mathcal{S}_C, B\in\mathcal{O}_C$ and all complex numbers λ. Moreover, the preannihilators of $\mathcal{S}_C$ and $\mathcal{O}_C$ were completely determined as follows.

Theorem 1.1 ([28, Theorem 3.1]). *Let C be a conjugation on $\mathcal{H}$. Then $(\mathcal{S}_C)_\perp=\mathcal{O}_C\cap\mathcal{B}_1(\mathcal{H})$ and $(\mathcal{O}_C)_\perp=\mathcal{S}_C\cap\mathcal{B}_1(\mathcal{H})$, where $\mathcal{B}_1(\mathcal{H})$ is the set of all trace class operators on $\mathcal{H}$.*

Recall that the preannihilator of a subspace $\mathcal{V}$ of $\mathcal{B}(\mathcal{H})$ is defined as

$$\mathcal{V}_\perp:=\big\{X\in\mathcal{B}_1(\mathcal{H}):\mathrm{tr}(AX)=0 \text{ for all } A\in\mathcal{V}\big\},$$

where $\mathrm{tr}(\cdot)$ denotes trace.

Inspired by the preceding results, we aim to further explore the connection between $\mathcal{S}_C$ and $\mathcal{O}_C$ for C a conjugation on $\mathcal{H}$. The present work is mainly motivated by the following observation:

$$A,B\in\mathcal{S}_C \implies [A,B]:=AB-BA\in\mathcal{O}_C. \tag{1.1}$$

If we denote by $[\mathcal{S}_C,\mathcal{S}_C]$ the collection of commutators of operators in $\mathcal{S}_C$, that is, $[\mathcal{S}_C,\mathcal{S}_C]=\{[A,B]:A,B\in\mathcal{S}_C\}$, then the preceding implication (1.1) exactly means that $[\mathcal{S}_C,\mathcal{S}_C]\subset\mathcal{O}_C$. Then a natural questions arises: Can one use operators in $[\mathcal{S}_C,\mathcal{S}_C]$ to represent every operator in $\mathcal{O}_C$?

In this paper, we shall prove the following result, which gives an answer to the preceding question.

Theorem 1.2. *If C is a conjugation on $\mathcal{H}$, then $\mathcal{O}_C=[\mathcal{S}_C,\mathcal{S}_C]+[\mathcal{S}_C,\mathcal{S}_C]$.*

Next let us review some results on the problem of determining which operators can be written as commutators of operators in $\mathcal{B}(\mathcal{H})$. In the case that $\dim\mathcal{H}<\infty$, it was proved in [1] that an operator $T\in\mathcal{B}(\mathcal{H})$ is a commutator of two operators in $\mathcal{B}(\mathcal{H})$ if and only if $\mathrm{tr}\,T=0$. In the case that $\dim\mathcal{H}=\infty$, Brown and Pearcy [5] proved that an operator $T\in\mathcal{B}(\mathcal{H})$ is a commutator of two operators in $\mathcal{B}(\mathcal{H})$ if and only if $T\neq\lambda I+K$ for any $\lambda\in\mathbb{C}\setminus\{0\}$ and any compact operator K.

Note that there exists no operator T in $\mathcal{O}_C$ such that $T=\lambda I+K$ for some $\lambda\in\mathbb{C}\setminus\{0\}$ and some compact operator K; moreover, $\mathrm{tr}\,T=0$ if $T\in\mathcal{O}_C\cap\mathcal{B}_1(\mathcal{H})$. Thus, $\mathcal{O}_C\subset[\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{H})]$ for any complex Hilbert space $\mathcal{H}$. It is natural to ask whether the equality $\mathcal{O}_C=[\mathcal{S}_C,\mathcal{S}_C]$ holds.

In the sequel, we shall prove that $[\mathcal{S}_C,\mathcal{S}_C]$ includes all finite-rank operators in $\mathcal{O}_C$ (see Lemma 2.2). It follows that $\mathcal{O}_C=[\mathcal{S}_C,\mathcal{S}_C]$ provided $\dim\mathcal{H}<\infty$ (see Corollary 2.2). However, we do not know whether this holds for any separable complex Hilbert space $\mathcal{H}$. Our proof of Theorem 1.2 depends on the result that $[\mathcal{S}_C,\mathcal{S}_C]$ contains all normal operators in $\mathcal{O}_C$ (see the proof of Theorem 1.2).

2 Proof of main result

We first make some preparations for the proof of Theorem 1.2.

Lemma 2.1. *Let C be a conjugation on $\mathcal{H}$. If $T\in\mathcal{O}_C$ is a normal operator with* card $\sigma(T)=2$, *then there exist $D,E\in\mathcal{S}_C$ with $\|D\|=1$ and $\|E\|=\|T\|$ such that $T=DE-ED$.*

Proof. Note that $CTC=-T^*$. We have $C(T-\lambda)C=-(T+\lambda)^*$ for all $\lambda\in\mathbb{C}$. Hence, $\lambda\in\sigma(T)$ if and only if $-\overline{\lambda}\in\sigma(T^*)$, which is equivalent to $-\lambda\in\sigma(T)$. Since card $\sigma(T)=2$, we can find nonzero $z_0\in\mathbb{C}$ such that $\sigma(T)=\{z_0,-z_0\}$. Then $\sigma(T/z_0)=\{1,-1\}$. Denote $S=T/z_0$. It suffices to prove the desired result for S.

Since S is still normal, we deduce that

$$S=\begin{bmatrix} I_1 & 0 \\ 0 & -I_2 \end{bmatrix}\begin{matrix}\ker(S-1)\\ \ker(S+1)\end{matrix},$$

where I_1 is the identity operator on $\ker(S-1)$ and I_2 is the identity operator on $\ker(S+1)$.

Note that $C(S^*+1)C=-(S-1)$. It follows that

$$C\big(\ker(S-1)\big)=\ker(S^*+1)=\ker(S+1)$$

and hence

$$C\big(\ker(S+1)\big)=\ker(S-1).$$

Then C can be written as

$$C=\begin{bmatrix} 0 & C_2 \\ C_1 & 0 \end{bmatrix}\begin{matrix}\ker(S-1)\\ \ker(S+1)\end{matrix}.$$

From $C^2=I$ (the identity operator on $\mathcal{H}$), one can see that $C_1:\ker(S-1)\to\ker(S+1)$ is conjugate-linear, invertible and $C_2=C_1^{-1}$. Also, note that $\dim\ker(S-1)=\dim\ker(S+1)$. For convenience, we denote $\mathcal{H}_1=\ker(S-1)$ and $\mathcal{H}_2=\ker(S+1)$.

Claim. There exists $D_1\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)$ such that $C_1D_1C_1=D_1^*$.

Choose a conjugation C_3 on $\mathcal{H}_2$ and set $D_1 = C_1^{-1}C_3$. Then $D_1 \in \mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)$ is unitary and $D_1^* = C_3C_1$. One can verify that

$$C_1D_1C_1 = C_1\big(C_1^{-1}C_3\big)C_1 = C_3C_1 = D_1^*.$$

Set

$$D = \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Then $D \in \mathcal{B}(\mathcal{H})$ and one can verify that $CDC = D^*$, that is, $D \in \mathcal{S}_C$. Furthermore, one can check that $[D,D^*] = S$. Set $E = D^*$. Then D and E satisfy all requirements. The proof is complete. □

Corollary 2.1. *Let C be a conjugation on $\mathcal{H}$. If $T \in \mathcal{O}_C$ is a diagonal operator, then there exist $D,E \in \mathcal{S}_C$ with $\|D\| = 1$ and $\|E\| = \|T\|$ such that $T = DE - ED$.*

Proof. Note that $CTC = -T^*$. We have $C(T-\lambda)C = -(T+\lambda)^*$ for all $\lambda \in \mathbb{C}$. Hence, $C(\ker\,(T+\lambda)^*) = \ker\,(T-\lambda)$. It follows that $\lambda \in \sigma_p(T)$ if and only if $-\overline{\lambda} \in \sigma_p(T^*)$ if and only if $-\lambda \in \sigma_p(T)$. Hence, without loss of generality, we may assume that

$$\sigma_p(T)\setminus\{0\} = \{\pm\lambda_i : i = 1,2,3,\ldots\}.$$

Note that T is diagonal. It follows that

$$\mathcal{H} = \ker T \oplus \left(\bigoplus_{i=1}^{\infty} \ker(T-\lambda_i)\right) \oplus \left(\bigoplus_{i=1}^{\infty} \ker(T+\lambda_i)\right).$$

Denote $\mathcal{H}_0 = \ker T$ and $\mathcal{H}_i = \ker(T-\lambda_i) \oplus \ker(T+\lambda_i)$ for each $i \geq 1$. Then $\mathcal{H} = \oplus_{i=0}^{\infty}\mathcal{H}_i$. Clearly, each $\mathcal{H}_i$ reduces T. Denote $T_i = T|_{\mathcal{H}_i}$, $i = 0,1,2,\ldots$.

Since

$$C\big(\ker\,(T+\lambda_i)\big) = \ker\,(T-\lambda_i)^* = \ker\,(T-\lambda_i),$$

it follows that

$$C\big(\ker\,(T-\lambda_i)\big) = \ker\,(T+\lambda_i)$$

and $C(\mathcal{H}_i) = \mathcal{H}_i$. Hence, each $\mathcal{H}_i$ reduces C. Also note that $\mathcal{H}_0$ reduces C. Denote $C_0 = C|_{\ker T}$ and $C_i = C|_{\mathcal{H}_i}$ for all $i \geq 1$. Then C_i is a conjugation on $\mathcal{H}_i$, $i = 0,1,2,\ldots$. From $CTC = -T^*$, one can see that $C_iT_iC_i = T_i^*$, $i = 0,1,2,\ldots$.

Note that $T|_{\mathcal{H}_0} = 0$ and $\sigma(T|_{\mathcal{H}_i}) = \{\lambda_i, -\lambda_i\}$ for each $i \geq 1$. Since T_i is normal, by Lemma 2.1, we can find $D_i, E_i \in \mathcal{S}_{C_i}$ with $\|D_i\| = 1$ and $\|E_i\| = \|T_i\|$ such that $T_i = [D_i, E_i]$ for all $i = 0,1,2,\ldots$.

Set $D = \oplus_{i=0}^{\infty} D_i$ and $E = \oplus_{i=0}^{\infty} E_i$. Then one can check that $D,E \in \mathcal{S}_C$, $\|D\| = 1$ and $\|E\| = \|T\|$, moreover, $T = DE - ED$. □

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Clearly, $\mathcal{O}_C$ is a linear subspace of $\mathcal{B}(\mathcal{H})$ and $[\mathcal{S}_C,\mathcal{S}_C]\subset\mathcal{O}_C$. Then $[\mathcal{S}_C,\mathcal{S}_C]+[\mathcal{S}_C,\mathcal{S}_C]\subset\mathcal{O}_C$. It suffices to prove $\mathcal{O}_C\subset[\mathcal{S}_C,\mathcal{S}_C]+[\mathcal{S}_C,\mathcal{S}_C]$.

Note that each operator $T\in\mathcal{O}_C$ is the sum of two normal operators in $\mathcal{O}_C$. In fact, if $T\in\mathcal{O}_C$, then one can check that $T_1:=(T+T^*)/2, T_2:=(T-T^*)/2$ are normal operators, $T_1,T_2\in\mathcal{O}_C$ and $T=T_1+T_2$. Hence, it suffices to prove the following claim.

Claim. $[\mathcal{S}_C,\mathcal{S}_C]$ contains all normal operators in $\mathcal{O}_C$.

Choose a normal operator $N\in\mathcal{O}_C$. We shall show that $N\in[\mathcal{S}_C,\mathcal{S}_C]$. We let $E_N(\cdot)$ denote the projection-valued spectral measure corresponding to N. Denote $\Sigma=\{\alpha\in\mathbb{C}:\text{Im }\alpha>0\}\cup\{\alpha\in\mathbb{C}:\text{Im }\alpha=0,\text{Re }\alpha>0\}$. Since $CNC=-N^*$, by [21, Theorem 2.2], it follows that

$$C\big(\text{ran }E_N(\sigma)\big)=\text{ran }E_N(-\sigma),$$

where $\sigma=\Sigma\cap\sigma(N)$ and $-\sigma=\{z\in\mathbb{C}:-z\in\sigma\}$. Furthermore,

$$C\big(\text{ran }E_N(-\sigma)\big)=\text{ran }E_N(\sigma).$$

Note that

$$\begin{aligned}&C(\ker N)=\ker N^*=\ker N,\\&\mathcal{H}=\ker N\oplus E_N(\sigma)\oplus E_N(-\sigma).\end{aligned}$$

Then N and C can be written as

$$N=\begin{bmatrix}0&0&0\\0&N_1&0\\0&0&N_2\end{bmatrix}\begin{matrix}\ker N\\E_N(\sigma)\\E_N(-\sigma)\end{matrix},\quad C=\begin{bmatrix}C_0&0&0\\0&0&C_2\\0&C_1&0\end{bmatrix}\begin{matrix}\ker N\\E_N(\sigma)\\E_N(-\sigma)\end{matrix}.$$

Then C_0 is a conjugation on $\ker N$, and both $C_1:E_N(\sigma)\longrightarrow E_N(-\sigma)$ and $C_2: E_N(-\sigma)\longrightarrow E_N(\sigma)$ are conjugate-linear, surjective isometries. From $C=C^{-1}$, one can see that $C_2=C_1^{-1}$. Also, one can see from $CN^*C=-N$ that $C_2N_2C_1=-N_1^*$. Thus, $N_2=-C_1N_1^*C_2=-C_1N_1^*C_1^{-1}$.

The rest of the proof is divided into two cases. If N_1 is a diagonal operator, then so is N_2, which implies that N is diagonal. By Corollary 2.1, $N\in[\mathcal{S}_C,\mathcal{S}_C]$. Next we consider the case that N_1 is not a diagonal operator. Since N_1 is normal, we deduce that $N_1\neq\lambda I+K$ for any $\lambda\in\mathbb{C}\setminus\{0\}$ and any compact K. By [5], there exist $\tilde{D},\tilde{E}\in\mathcal{B}(E_N(\sigma))$ such that $[\tilde{D},\tilde{E}]=N_1$.

Set

$$D=\begin{bmatrix}0&0&0\\0&\tilde{D}&0\\0&0&C_1\tilde{D}^*C_2\end{bmatrix}\begin{matrix}\ker N\\E_N(\sigma)\\E_N(-\sigma)\end{matrix},\quad E=\begin{bmatrix}0&0&0\\0&\tilde{E}&0\\0&0&C_1\tilde{E}^*C_2\end{bmatrix}\begin{matrix}\ker N\\E_N(\sigma)\\E_N(-\sigma)\end{matrix}.$$

Direct calculations show that $D,E\in\mathcal{S}_C$ and

$$[C_1\tilde{D}^*C_2,C_1\tilde{E}^*C_2]=C_1[\tilde{D}^*,\tilde{E}^*]C_2=-C_1[\tilde{D},\tilde{E}]^*C_2=-C_1N_1^*C_2.$$

Hence,

$$[D,E]=0\oplus N_1\oplus(-C_1N_1^*C_2)=0\oplus N_1\oplus N_2=N.$$

This completes the proof. □

Lemma 2.2. *Let C be a conjugation on $\mathcal{H}$. If $T\in\mathcal{O}_C$ is of finite rank, then $T\in[\mathcal{S}_C,\mathcal{S}_C]$.*

Proof. Set $\mathcal{M}=\text{ran }T+\text{ran }T^*$. Clearly, $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ and $\mathcal{M}$ reduces T. Also one can check that $C(\mathcal{M})\subset\mathcal{M}$ and $C(\mathcal{M}^\perp)\subset\mathcal{M}^\perp$. Thus, T and C can be written as

$$T=\begin{bmatrix}A&0\\0&0\end{bmatrix}\begin{matrix}\mathcal{M}\\\mathcal{M}^\perp\end{matrix},\quad C=\begin{bmatrix}C_0&0\\0&C_1\end{bmatrix}\begin{matrix}\mathcal{M}\\\mathcal{M}^\perp\end{matrix},$$

where C_0 is a conjugation on $\mathcal{M}$ and C_1 is a conjugation on $\mathcal{M}^\perp$. Note that $C_0AC_0=-A^*$.

Suppose that $\dim\mathcal{M}=n$ and we can choose an orthonormal basis $\{e_1,e_2,\cdots,e_n\}$ of $\mathcal{M}$ such that $C_0e_i=e_i$, $i=1,2,\ldots,n$ (see [16]). We assume relative to $\{e_1,e_2,\cdots,e_n\}$, A can be written as

$$A=\begin{bmatrix}a_{1,1}&a_{1,2}&\cdots&a_{1,n}\\a_{2,1}&a_{2,2}&\cdots&a_{2,n}\\\vdots&\vdots&\ddots&\vdots\\a_{n,1}&a_{n,2}&\cdots&a_{n,n}\end{bmatrix}\begin{matrix}e_1\\e_2\\\vdots\\e_n\end{matrix}.$$

For $i,j\in\{1,2,\ldots,n\}$, note that

$$\begin{aligned}a_{j,i}&=\langle Ae_i,e_j\rangle=\langle -C_0A^*C_0e_i,e_j\rangle=\langle -C_0A^*e_i,e_j\rangle\\&=-\langle C_0e_j,A^*e_i\rangle=-\langle e_j,A^*e_i\rangle\\&=-\langle Ae_j,e_i\rangle=-a_{i,j}.\end{aligned}$$

We define

$$c_{i,j}=\begin{cases}\dfrac{a_{i,j}}{i-j}, & i\neq j,\\ 0, & i=j.\end{cases}$$

Hence, $c_{i,j}=c_{j,i}$ for $i,j\in\{1,2,\ldots,n\}$. We define $D_1,E_1\in\mathcal{B}(\mathcal{M})$ as

$$D_1=\begin{bmatrix}1&&&\\&2&&\\&&\ddots&\\&&&n\end{bmatrix}\begin{matrix}e_1\\e_2\\\vdots\\e_n\end{matrix},\quad E_1=\begin{bmatrix}c_{1,1}&c_{1,2}&\cdots&c_{1,n}\\c_{2,1}&c_{2,2}&\cdots&c_{2,n}\\\vdots&\vdots&\ddots&\vdots\\c_{n,1}&c_{n,2}&\cdots&c_{n,n}\end{bmatrix}\begin{matrix}e_1\\e_2\\\vdots\\e_n\end{matrix}.$$

One can verify that $C_0D_1C_0=D_1^*$ and $C_0E_1C_0=E_1^*$, moreover, $A=D_1E_1-E_1D_1$.

Define

$$D=\begin{bmatrix}D_1&0\\0&0\end{bmatrix}\begin{matrix}\mathcal{M}\\\mathcal{M}^\perp\end{matrix},\quad E=\begin{bmatrix}E_1&0\\0&0\end{bmatrix}\begin{matrix}\mathcal{M}\\\mathcal{M}^\perp\end{matrix}.$$

Clearly, $D,E\in\mathcal{S}_C$. Also, one can check that $T=DE-ED$. $\square$

Corollary 2.2. *Let C be a conjugation on $\mathcal{H}$. If $\dim\mathcal{H}<\infty$, then $\mathcal{O}_C=[\mathcal{S}_C,\mathcal{S}_C]$.*

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