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Reduced Basis Method Based on Fourier Transform for Time-Dependent Parameterized Nonlocal Problems

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Abstract. In the paper, a reduced basis (RB) method for time-dependent nonlocal problems with a special parameterized fractional Laplace kernel function is proposed. Because of the lack of sparsity of discretized nonlocal systems compared to corresponding local partial differential equation (PDE) systems, model reduction for nonlocal systems becomes more critical. The method of snapshots and greedy (MOS-greedy) algorithm of RB method is developed for nonlocal problems with random inputs, which provides an efficient and reliable approximation of the solution. A major challenge lies in the excessive influence of the time domain on the model reduction process. To address this, the Fourier transform is applied to convert the original time-dependent parabolic equation into a frequency-dependent elliptic equation, where variable frequencies are independent. This enables parallel computation for approximating the solution in the frequency domain. Finally, the proposed MOS-greedy algorithm is applied to the nonlocal diffusion problems. Numerical results demonstrate that it provides an accurate approximation of the full order problems and significantly improves computational efficiency.

AMS subject classifications: 65N99, 60H35, 35R60
Key words: Nonlocal problems, reduced basis method, method of snapshots, greedy algorithm, Fourier transform.

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1 Introduction

Historically, continuum models were predominantly described by partial differential equations (PDEs) based on local information. Later, the study of complex systems with singularities and anomalies, as well as those involving nonlocal interactions, became the focus. Nonlocal equations have been shown to provide significantly better models than their local counterparts in various applications. Examples include optimal control problems involving the Bellman equation derived from Levy processes, denoising models in nonlocal image processing [4,26], and particle systems modeling the nonlocal porous medium equation, the Hamilton-Jacobi equation with fractional diffusion [10,27], and conservation laws with fractional diffusion [7], among others.

Initial research on nonlocal models focused primarily on scalar problems [26, 30], with applications in image processing and steady-state diffusion, respectively. Subsequently, Du *et al.* established a more systematic mathematical framework for nonlocal problems parallel to classical local PDEs. They developed the nonlocal vector calculus [15, 16], and extensive research followed on functional analysis of nonlocal spaces, operators, and calculus of variations [21, 32, 41, 42]. Since exact solutions to nonlocal models are generally unavailable, numerical solutions posed new challenges for algorithm development and numerical analysis. This necessitated the development of robust and adaptive algorithms, as well as various numerical approximation schemes for nonlocal models [18–20]. Various applications of nonlocal models and connections to existing mathematical studies and numerical techniques have enabled nonlocal modeling to bridge the gap between multiscale modeling, analysis, and simulation [13,17,43]. Further rigorous mathematical analysis of nonlocal models was provided in [2,22]. In recent years, nonlocal models have been used in many areas, such as phase transition [2, 23], nonlocal peridynamic models [1,40], nonlocal dispersal models [6,11] and option pricing in models with jumps [38].

Although nonlocal modeling can complement or replace traditional local modeling approaches based on PDEs, a priori for the value of the parameters in nonlocal kernel functions is unknown in practical scenarios involving modeling and prediction. In such case, an approximate solution is required not only as a function of a spatial variable but also as a function of model parameters. Compared to local models, the coefficient matrix of the discrete system for nonlocal models is typically dense, leading to higher computational costs. Therefore, reduced order models (ROMs) are necessary to approximate solutions efficiently, reducing computational costs. In this paper, the primary focus and challenge lie in selecting an appropriate model reduction method and accurately capturing the essential