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On the Long-Time H^1-Stability of the Linearly Extrapolated BDF2 Time-Stepping Scheme for Coupled Multiphysics Flow Problems

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Abstract. The purpose of the current article is to study the H^1-stability for all positive time of the linearly extrapolated BDF2 time-stepping scheme for the magnetohydrodynamics and Boussinesq equations. Specifically, we discretize in time using the linearly backward differentiation formula, and by employing both the discrete Gronwall lemma and the discrete uniform Gronwall lemma, we establish that each numerical scheme is uniformly bounded in the H^1-norm.

AMS subject classifications: 35Q56, 35Q35

Key words: Magnetohydrodynamics equations, Boussinesq equations, linearly extrapolated BDF2 timestepping scheme, long-time stability.

1 Introduction

Ensuring long-term stability of a scheme is crucial for the accuracy of long time simulations, and this aspect has been extensively studied. For instance, in [2,4,11, 13,14], the authors demonstrated the H^1-stability of various schemes for the two-dimensional Navier-Stokes equations. In [3, 9], the authors not only established

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the H^1-stability of the implicit Euler scheme for a homogeneous two-phase flow model and for the 2D thermohydraulics equations, respectively, but also demonstrated that the global attractors of the numerical scheme converge to the global attractor of the continuous system as the time step approaches zero. In [12], the authors examined two-dimensional magnetohydrodynamics (MHD) equations and demonstrated that the implicit Euler scheme is uniformly stable in time with respect to the H^2-norm. In this article, we re-examine the 2D MHD equations and Boussinesq equations by discretizing them in time with the linearly extrapolated BDF2 timestepping scheme. We demonstrate that these schemes are uniformly stable in the H^1-norm.

The BDF2 temporal discretization is a popular time-stepping method in simulations because of its second-order accuracy and appealing stability characteristics. In [1], the authors proved the L^2- stability of the linearly extrapolated BDF2 timestepping scheme for Navier-Stokes, Boussinesq, and magnetohydrodynamics equations. Recently, Rebholz and Tone have expanded the L^2- stability result for the 2D NSE to the H^1-norm (see [7]). In this article, our goal is to expand the result in [1] and show the H^1-stability of the BDF2 scheme for the two-dimensional magnetohydrodynamics and Boussinesq equations. We first demonstrate that the schemes are uniformly bounded in L^2, and then use these results to establish uniform boundedness in H^1.

This article is organized as follows. In Section 2 we present some mathematical preliminaries. In Sections 3 and 4 we prove the H^1-stability for the magnetohydrodynamics and the Boussinesq equations, respectively. Section 5 gives the conclusion of the paper.

2 Mathematical preliminaries

In this article, we assume that $\Omega\subset\mathbb{R}^2$ is an open bounded domain with boundary $\partial\Omega$ of class C^2. We denote $L^2(\Omega)$-inner product and norm by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The analysis presented in the paper uses G-matrix and its associated G-norm together with some properties, see [5]. We define the G-matrix and its associated G-norm as follows:

$$G=\begin{bmatrix} 1/2 & -1 \\ -1 & 5/2 \end{bmatrix},$$

and

$$\|x\|_G^2=(x,Gx)=\frac{\|b\|^2+\|2b-a\|^2}{2} \quad \text{for any} \quad x=[a,b]^T\in\mathbb{R}^2,$$