Commun. Math. Res.
doi: 10.4208/cmr.2025-0016

Vol. **41**, No. 2, pp. 148-172
June 2025

Homoclinic Solutions for a Class of Hamiltonian Systems with Small External Perturbations

Wenzhuang Zhu[1], Chunhui Hu[2,*] and Shuguan Ji[3]

[1] *School of Mathematics and LPMC, Nankai University, Tianjin 300071, P.R. China.*
[2] *School of Mathematics, Nankai University, Tianjin 300071, P.R. China.*
[3] *School of Mathematics and Statistics and Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun 130024, P.R. China.*

Received 21 March 2025; Accepted 6 May 2025

Abstract. This paper is concerned with the existence of nontrivial homoclinic solutions for a class of second order Hamiltonian systems with external forcing perturbations $\ddot{q}+A\dot{q}+V_q(t,q)=f(t)$, where $q=(q_1,q_2,\cdots,q_N)\in\mathbb{R}^N$, A is an antisymmetric constant $N\times N$ matrix, $V(t,q)=-K(t,q)+W(t,q)$ with $K,W\in C^1(\mathbb{R},\mathbb{R}^N)$ and satisfying $b_1|q|^2\le K(t,q)\le b_2|q|^2$ for some positive constants $b_2\ge b_1>0$ and external forcing term $f\in C(\mathbb{R},\mathbb{R}^N)$ being small enough. Under some new weak superquadratic conditions for W, by using the mountain pass theorem, we obtain the existence of at least one nontrivial homoclinic solution.

AMS subject classifications: 58E05, 58E50, 34C37

Key words: Homoclinic solution, Hamiltonian system, mountain pass theorem, critical point.

1 Introduction and main results

In this paper, we consider a class of second order Hamiltonian systems

$$\ddot{q}+A\dot{q}+V_q(t,q)=f(t), \tag{1.1}$$

*Corresponding author. *Email addresses:* `zhuwz76@nankai.edu.cn` (W. Zhu), `hchui2022@163.com` (C. Hu), `jishuguan@hotmail.com` (S. Ji)

where $t\in\mathbb{R}$, $q=(q_1,q_2,\cdots,q_N)\in\mathbb{R}^N$, A is an antisymmetric constant $N\times N$ matrix, $V\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$,

$$V_q(t,q)=\left(\frac{\partial V}{\partial q_1}(t,q),\frac{\partial V}{\partial q_2}(t,q),\cdots,\frac{\partial V}{\partial q_N}(t,q)\right)$$

and $f\in C(\mathbb{R},\mathbb{R}^N)$. As usual, a solution $q(t)$ of (1.1) is called a nontrivial homoclinic (to 0) if $q(t)\neq 0$, $q(t)\to 0$ and $\dot{q}(t)\to 0$ as $t\to\pm\infty$.

The existence of homoclinic solutions of Hamiltonian systems is a classical problem, and it has been extensively studied since Poincaré [11] recognized its importance in dynamical systems at the end of 19th century. In last four decades, variational method has been successfully used to investigate the homoclinic solutions of Hamiltonian systems, see [2,5,6,8–10,12–14,16,17,21,22,24,25,27] and references therein. It is worth mentioning the celebrated work of Zelati and Rabinowitz [25], where they considered the classical Hamiltonian systems

$$\ddot{q}+L(t)q+W_q(t,q)=0, \tag{1.2}$$

which corresponds the special case of (1.1) in which A and f are vanishing (i.e., $A\equiv 0$, $f\equiv 0$) and $V=-L(t)q\cdot q/2+W(t,q)$ for some symmetric matrix valued function $L\in C(\mathbb{R},\mathbb{R}^{N\times N})$. By using a variational method in virtue of the famous mountain pass technique [24], they established the existence of infinitely many homoclinic orbits of (1.2) under the Ambrosetti-Rabinowitz superquadratic condition, that is, there is a $\mu>2$ such that

$$0<\mu W(t,q)\leq W_q(t,q)\cdot q,\quad \forall q\in\mathbb{R}^N\setminus\{0\}. \tag{1.3}$$

Since then this superquadratic condition (1.3) was used in most of the research on homoclinic solutions of Hamiltonian systems. Recently, some authors considered to use some new superquadratic condition other than Ambrosetti-Rabinowitz superquadratic condition to ensure the existence of homoclinic solutions for the classical second order Hamiltonian systems (1.2) and first order Hamiltonian systems (see [3–5,7,9,10,12–14,16,17,19–23,26,27]). In [7], the authors studied the existence of ground state homoclinic solutions for system (1.2) with more general potentials under some monotone condition instead of the Ambrosetti-Rabinowitz superquadratic condition.

Izydorek and Janczewska [6] considered a more general system than (1.2), which has pure external force, that is

$$\ddot{q}+V_q(t,q)=f(t), \tag{1.4}$$

where $V(t,q)$ satisfies