Commun. Math. Res.
doi: 10.4208/cmr.2025-0017

Vol. **41**, No. 2, pp. 173-188
June 2025

The Variants of the MGSS Methods for Complex Symmetric Linear System of Equations

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Received 1 April 2025; Accepted 4 June 2025

Abstract. Recently, inspired by a modified generalized shift-splitting iteration method for complex symmetric linear systems, we propose two variants of the modified generalized shift-splitting iteration (MGSS) methods for solving complex symmetric linear systems. One is a parameterized MGSS iteration method and the other is a modified parameterized MGSS iteration method. We prove that the proposed methods are convergent under appropriate constraints on the parameters. In addition, we also give the eigenvalue distributions of different preconditioned matrices to verify the effectiveness of the preconditioners proposed in this paper.

AMS subject classifications: 15A24, 65F10

Key words: Complex symmetric linear system, variants of MGSS method, convergence analysis.

1 Introduction

In this paper, we consider complex symmetric linear systems. Such linear systems appears in eddy current problem [3], electrical power modeling [13], FFT-based

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solution of certain time-dependent PDEs [6], structural dynamics [12], molecular scattering [18], and many other applications. In order to avoid complex operation and construct effective preconditioners, one represent the complex-valued linear system as a real-valued form. For instance, complex symmetric linear system of the form

$$Au=(W+iT)u=b, \tag{1.1}$$

which can be expressed as an equivalent real matrix form

$$\mathcal{A}v=\begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} p \\ q \end{pmatrix}, \tag{1.2}$$

where $u=x+iy, b=p+iq$, $i=\sqrt{-1}, x,y,p,q\in\mathbb{R}^n$, $W\in\mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, T is a symmetric positive semidefinite matrix. The existence and uniqueness of the solution for the linear system (1.1) can refer to the book [5], which gives a detailed introduction for the solution.

We know that in order to solve complex symmetric linear systems more effectively, many iteration methods have been proposed. In 2010, Bai *et al.* [2] proposed a modified Hermitian and Skew-Hermitian splitting iteration method to solve the complex symmetric linear system. Theoretical analysis proved that MHSS method is unconditionally convergent, and numerical results shown that MHSS method performs better than HSS method in terms of iteration steps and CPU times. In order to further improve and accelerate the MHSS method, Bai *et al.* [3] studied a preconditioned variant of the MHSS method. The PMHSS method not only has a more general framework, but also has better numerical properties than the MHSS method. Subsequently, a series of methods such as LPMHSS [17], GPMHSS [15], a two-parameter GPMHSS [16], two variants of PMHSS [7] and APMHSS [27] were proposed one after another. In addition, there are more iteration methods for solving complex linear systems, which can be seen in [10, 14, 21, 23, 28] and references therein.

In recent years, many researchers have done a lot of work for the linear system (1.2) with a block 2×2 structure. For example, GSOR method [20], AGSOR method [11], PMHSS method [4, 19], ADPMHSS method [22], AOR-Uzawa method [8], DSS method [26], GSS method [9, 25], MGHSS method [1, 24], MTSS method [14], etc. Next, we list the iterative schemes of the GSS methods and MGSS method.

The GSS iteration method [9]: Given an initial guess u^0, for $k=0,1,2,\ldots$, until u^k converges, compute

$$\frac{1}{2}\begin{pmatrix} \widetilde{\alpha}I+W & -T \\ T & \widetilde{\beta}I+W \end{pmatrix}u^{k+1}=\frac{1}{2}\begin{pmatrix} \widetilde{\alpha}I-W & T \\ -T & \widetilde{\beta}I-W \end{pmatrix}u^k+\begin{pmatrix} p \\ q \end{pmatrix}. \tag{1.3}$$