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$\mathbf{H}(\mathrm{curl}^2)$ Conforming Element for Maxwell's Transmission Eigenvalue Problem Using Fixed-Point Approach

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Abstract. Using newly developed $\mathbf{H}(\mathrm{curl}^2)$ conforming elements, we solve the Maxwell's transmission eigenvalue problem. Both real and complex eigenvalues are considered. Based on the fixed-point weak formulation with reasonable assumptions, the optimal error estimates for numerical eigenvalues and eigenfunctions (in the $\mathbf{H}(\mathrm{curl}^2)$-norm and $\mathbf{H}(\mathrm{curl})$-semi-norm) are established. Numerical experiments are performed to verify the theoretical assumptions and confirm our theoretical analysis.

AMS subject classifications: 65N25, 65N30

Key words: Maxwell's transmission eigenvalues, curl-curl conforming element, error estimates.

1 Introduction

The transmission eigenvalue problem has important applications in the area of inverse scattering, e.g. simulating non-destructive test of anisotropic materials. For some background materials such as existence theory, application, and reconstruction of transmission eigenvalues, we refer readers to [5–8,11] and references therein. Naturally, numerical computation of transmission eigenvalues has attracted the attention of scientific community. There have been some research works on numerical methods for Helmholtz transmission eigenvalue problem (HTEP), see, e.g. [1,9,12,17,18,26,27]. However, numerical treatment of the Maxwell's transmission eigenvalue problem (MTEP) is relatively rare. An earlier work on the subject can be found in [19] where a curl-conforming and a mixed finite element were proposed. The authors reduced the MTEP to two coupled eigenvalue problems involving the second-order curl operator. Huang *et al.* [16] proposed an eigen-

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solver for computing a few smallest positive Maxwell's transmission eigenvalues. More recently An and Zhang [2] studied a spectral method for MTEP on spherical domains and obtained numerical eigenvalues with superior accuracy. The MTEP is a non-self-adjoint and non-elliptic problem involving the quad-curl operator, which makes the error analysis of its numerical methods difficult (see the concluding remark in [19]). There have been some related works on numerical methods for equations with the quad-curl operator and the associated eigenvalue problems [4, 13, 22, 24, 29, 30].

In the finite element error analysis for HTEP, the solution operator of its source problem is readily defined to guarantee its compactness in the solution space. However, it is difficult to define a compact solution operator for MTEP in $\mathbf{H}(\mathrm{curl}^2)$. Fortunately, the fixed-point weak formulation in [6–8, 20, 23] for MTEP leads to a source problem with a well-defined compact solution operator whose image space is also contained in $\mathbf{H}(\mathrm{div})$. The fixed-point weak formulation is a generalized eigenvalue problem with the eigenvalue as its parameter. To solve it numerically, an iterative method is usually adopted. An analysis framework of the iterative method for HTEP is well established in [21] and further developed in [25], which motivate us to use it for MTEP.

Recently, Zhang *et al.* [28] and Hu *et al.* [14, 15] proposed $\mathbf{H}(\mathrm{curl}^2)$-conforming (or curl-curl conforming) finite elements for solving partial differential equations with the quad-curl operator. In this paper, we use these newly developed $\mathbf{H}(\mathrm{curl}^2)$-conforming elements to solve MTEP in anisotropic inhomogeneous medium. Thanks to the conformity of the finite element space, it makes possible to establish convergence theory for the proposed method. We first prove the coercivity of bilinear form of the fixed-point weak formulation. Then we prove the uniform convergence of discrete operator in $\mathbf{H}_0(\mathrm{curl}^2,D)$. Under the assumption on the uniform lower bound (which can be verified numerically) of the discrete fixed-point function, the error estimate of discrete eigenvalue is proved using the Lagrange mean value theorem. Our analysis also includes the complex eigenvalue case with the fixed-point weak formulation being modified to guarantee the coercivity of the sesqui-linear form. To the best of our knowledge, this is the first numerical method with theoretical proof for MTEP with variable coefficients on general polygonal and polyhedral domains.

The rest of the paper is organized as follows. In Section 2, the fixed-point weak formulation and its curl-curl conforming element discretization is given then the Maxwell's transmission eigenvalue is expressed as the root of a fixed-point function. In Section 3, we discuss the error estimates for real eigenvalues. The solution operator and some associated discrete operators are defined and the compactness of the solution operator is stated. The optimal error estimates are proved using the approximation relations among discrete operators and Babuska-Osborn's theory. The error estimates for complex eigenvalues are proved in Section 4. Finally, in Section 5 we present several numerical examples with different indices of fraction to validate the assumption on the uniform lower bound of discrete fixed-point function and convergence order of curl-curl conforming element. The upper boundedness property of the real numerical eigenvalues is also verified in this section.