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Highly Efficient Gauss's Law-Preserving Spectral Algorithms for Maxwell's Double-Curl Source and Eigenvalue Problems Based on Eigen-Decomposition

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Abstract. In this paper, we present the Gauss's law-preserving spectral methods and their efficient solution algorithms for curl-curl source and eigenvalue problems arising from Maxwell's equations. Arbitrary order $\boldsymbol{H}(\text{curl})$-conforming spectral basis functions in two and three dimensions are firstly proposed using compact combination of Legendre polynomials. A mixed formulation involving a Lagrange multiplier is then adopted to preserve the Gauss's law in the weak sense. To overcome the bottleneck of computational efficiency caused by the saddle-point nature of the mixed scheme, we present highly efficient algorithms based on reordering and decoupling of the linear system and numerical eigen-decomposition of 1D mass matrix. The proposed solution algorithms are direct methods requiring only several matrix-matrix or matrix-tensor products of N-by-N matrices, where N is the highest polynomial order in each direction. Compared with other direct methods, the computational complexities are reduced from $\mathcal{O}(N^6)$ and $\mathcal{O}(N^9)$ to $\mathcal{O}(N^{\log_2 7})$ and $\mathcal{O}(N^{1+\log_2 7})$ with small and constant pre-factors for 2D and 3D cases, respectively. Moreover, these algorithms strictly obey the Helmholtz-Hodge decomposition, thus totally eliminate the spurious eigen-modes of non-physical zero eigenvalues for convex domains. Ample numerical examples for solving Maxwell's source and eigenvalue problems are presented to demonstrate the accuracy and efficiency of the proposed methods.

AMS subject classifications: 65N35, 65N22, 65N25, 65F05

Key words: Spectral method, time-harmonic Maxwell's equation, structure-preserving method, Gauss's law preservation, fast solver.

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1 Introduction

This work concerns accurate and efficient spectral methods for two typical model problems arising from time-harmonic Maxwell's equations in two and three dimensions, i.e. the double-curl source problem

$$\begin{aligned} &\nabla\times\nabla\times\boldsymbol{u}+\kappa\boldsymbol{u}=\boldsymbol{f}, \quad \nabla\cdot\boldsymbol{u}=\rho \quad &&\text{in } \Omega,\\ &\boldsymbol{n}\times\boldsymbol{u}=\boldsymbol{0} &&\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

and the eigenvalue problem: Find $\lambda\in\mathbb{R}$ and $\boldsymbol{u}\neq\boldsymbol{0}$ such that

$$\begin{aligned} &\nabla\times\nabla\times\boldsymbol{u}=\lambda\boldsymbol{u}, \quad \nabla\cdot\boldsymbol{u}=0 \quad &&\text{in } \Omega,\\ &\boldsymbol{n}\times\boldsymbol{u}=\boldsymbol{0} &&\text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Here, $\boldsymbol{x}\in\Omega\subset\mathbb{R}^d, d=2,3, \kappa\in\mathbb{C}, \boldsymbol{u}(\boldsymbol{x})$ usually represents the electric field, $\boldsymbol{f}(\boldsymbol{x})$ is the current density and $\rho(\boldsymbol{x})$ is the charge density satisfying $\rho=\nabla\cdot\boldsymbol{f}/\kappa$, $\boldsymbol{n}$ is the unit outward normal vector to the boundary $\partial\Omega$ and this boundary condition corresponds to the perfect electric conducting boundary condition. The Gauss's law $\nabla\cdot\boldsymbol{u}=\rho$ (respectively $\nabla\cdot\boldsymbol{u}=0$) is implicitly implied by the double-curl equation and serves as intrinsic constraint for the model problems.

Accurate numerical simulation of these double-curl problems with Gauss's law constraint plays a crucial role in understanding physical mechanisms and solving practical engineering problems related to electromagnetism, such as astrophysics, inertial confinement fusion, telecommunication, electromagnetic materials and devises, semiconductor manufacturing, quantum chromodynamics, quantum field theory and more [6, 7, 10, 19, 20, 25, 27, 31, 33, 41]. Nevertheless, the double-curl problems (1.1) and (1.2) are prohibitively difficult to solve numerically, due to the involvement of Gauss's law constraint. On the one hand, it poses significant challenges to construct suitable approximation spaces that guarantee the well-posedness of discrete problems and maintain the Gauss's law at the discrete level (either strongly or weakly). Failure to satisfy this constraint may result in severe numerical inaccuracies or the occurrence of spurious solutions. On the other hand, the saddle-point nature induced by the constraint makes them more expensive to solve and more challenging to construct efficient solution algorithms, compared with their self-adjoint coercive counterparts.

These challenges and difficulties have attracted increasing interest from the community in the past six decades, and a great many studies have been devoted to solving the Maxwell's double-curl problem and associated eigenvalue problem numerically. We restrict our review of literature to methods based on weak or variational formulation, which leaves out a vast of algorithms based on staggered-grid finite difference Yee scheme (see, e.g. [37, 38]) or divergence-correction technique (see, e.g. [2, 11, 28]). The endeavors trace back to Nédélec [29, 30], who firstly proposed two families of edge elements to discretize the Maxwell's equation (also comprehensive discussions in [26, 36]). Kikuchi [21] proposed a mixed formulation by introducing a Lagrange multiplier to