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New Finite Volume Element Schemes Based on a Two-Layer Dual Strategy

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Abstract. A two-layer dual strategy is proposed in this work to construct a new family of high-order finite volume element (FVE-2L) schemes that can avoid main common drawbacks of the existing high-order finite volume element (FVE) schemes. The existing high-order FVE schemes are complicated to construct since the number of the dual elements in each primary element used in their construction increases with a rate $\mathcal{O}((k+1)^2)$, where k is the order of the scheme. Moreover, all k-th-order FVE schemes require a higher regularity H^{k+2} than the approximation theory for the L^2 theory. Furthermore, all FVE schemes lose local conservation properties over boundary dual elements when dealing with Dirichlet boundary conditions. The proposed FVE-2L schemes has a much simpler construction since they have a fixed number (four) of dual elements in each primary element. They also reduce the regularity requirement for the L^2 theory to H^{k+1} and preserve the local conservation law on all dual elements of the second dual layer for both flux and equation forms. Their stability and H^1 and L^2 convergence are proved. Numerical results are presented to illustrate the convergence and conservation properties of the FVE-2L schemes. Moreover, the condition number of the stiffness matrix of the FVE-2L schemes for the Laplacian operator is shown to have the same growth rate as those for the existing FVE and finite element schemes.

AMS subject classifications: 65N08, 65N12, 65N30
Key words: Finite volume, two-layer dual mesh, conservation, L^2 estimate, minimum angle condition.

1 Introduction

The finite volume element method [2–5,7,10,12,14,16,18,19,24,26,28], also known as the generalized difference method, is a type of the finite volume method [12,15,17,25,32–34,

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38,39,41] that approximates the solution of partial differential equations (PDEs) in a finite element space. It inherits many advantages of both the finite element method, such as a straightforward definition of the gradient, and the finite volume method, such as the famous local conservation law. Till now, much progress has been made in the algorithmic development [7,9,16,19,30], stability analysis and H^1 estimation [9–11,20,27,32,39,40], L^2 estimation [7,8,21,22,30,35–37], and superconvergence analysis [6,29,31]. Nevertheless, there are still some open issues that have to be addressed.

Firstly, it is still complicated to construct high-order FVE schemes. Like other finite volume (FV) schemes, FVE schemes form their approximation equations by integrating the weak formulation of the underlying partial differential equations over dual elements. A commonly used strategy in FVE schemes is to define a dual element around each degree of freedom for Lagrange-type schemes. Thus, for each primary element, there are $(k+1)(k+2)/2$ dual elements for a k-th-order FVE scheme over triangular meshes [9,30,32] and $(k+1)^2$ dual elements for a bi-k-th-order FVE scheme over quadrilateral meshes [22,37,39]. It becomes increasingly complicated and computationally burdensome to partition a primary into so many dual regions even increasing k to 3 and 4.

Secondly, existing high-order FVE schemes require a higher regularity ($u\in H^{k+2}$) than what is needed for function approximation ($u\in H^{k+1}$) for the L^2 theory. The optimal L^2 convergence rate of a FVE scheme depends on the choice of the dual strategy. A unified L^2 analysis for FVE schemes on quadrilateral meshes has been provided in [21] by establishing some numerical quadrature equivalence, and the L^2 result for high-order FVE schemes on triangular meshes has been proved in [30] by proposing an orthogonality condition. Some other L^2 results for high-order FVE schemes can be found in [22,37]. However, all of the above L^2 results require $u\in H^{k+2}$ for k-th-order ($k\geq 2$) FVE schemes, which is a higher regularity requirement than $u\in H^{k+1}$ of the approximation theory.

Thirdly, Dirichlet boundary conditions may disrupt the conservation property on boundary dual elements in existing FVE schemes. To illustrate this, we take the following elliptic boundary value problem (BVP) on a bounded polygonal domain $\Omega\subset\mathbb{R}^2$ as an example:

$$\begin{cases} -\nabla\cdot(\mathbb{D}\nabla u)=f & \text{in } \Omega, \qquad (1.1\text{a})\\ u=0 & \text{on } \partial\Omega, \qquad (1.1\text{b})\end{cases}$$

where $f\in L^2(\Omega)$, and the diffusion tensor $\mathbb{D}=(d_{ij})_{i,j=1,2}$ is bounded by

$$\gamma_1(\xi,\xi)\leq(\mathbb{D}\xi,\xi)\leq\gamma_2(\xi,\xi),\quad \forall\xi\in\mathbb{R}^2,$$

and γ_1 and γ_2 are positive constants. Integrating the Eq. (1.1a) over a dual element K^*, one has the local conservation law in equation form (with discretization) given by

$$-\iint_{K^*}\nabla\cdot(\mathbb{D}\nabla u_h)\,\mathrm{d}x\mathrm{d}y=\iint_{K^*}f\,\mathrm{d}x\mathrm{d}y, \tag{1.2}$$