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Global-in-Time Energy Stability for a General Class of Stabilization Single-Step Schemes Applied to the Swift-Hohenberg Equation

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Abstract. The linear stabilization approach is well-known for facilitating the use of large time steps in solving gradient flows while maintaining stability. However, the up-to-date analysis of energy stability relies on either a global Lipschitz nonlinearity or an ℓ^∞ bound assumption of numerical solutions. Considering the Swift-Hohenberg equation that lacks a global Lipschitz nonlinearity, we develop a unified framework to analyze the energy stability and characterize the stabilization size for a class of single-step schemes employing spatial Fourier pseudo-spectral discretization. First, assuming that all stage solutions are bounded in the ℓ^∞ norm, we illustrate that the energy obtained from a single-step scheme with non-negative energy-stability-preserving coefficient is unconditionally dissipative, as long as a sufficiently large stabilization parameter is employed. To justify the ℓ^∞ bound assumption of solutions, we use the third-order exponential-time-differencing Runge-Kutta scheme as a case study to establish a uniform-in-time discrete H^2 bound for stage solutions under an $\mathcal{O}(1)$ time step constraint. This leads to a uniform ℓ^∞ bound of stage solutions through discrete Sobolev embedding. Consequently, we achieve a stabilization parameter of $\mathcal{O}(1)$, which is independent of the time step, thereby ensuring the energy stability. The global-in-time energy stability analysis and characterization of the stabilization parameter represent significant advancements for general single-step schemes applied to a gradient flow without the global Lipschitz continuity.

AMS subject classifications: 35Q92, 35B36, 65L06

Key words: Swift-Hohenberg equation, single-step scheme, energy-stability-preserving coefficient, global-in-time energy stability.

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1 Introduction

In this paper, we consider the Swift-Hohenberg (SH) equation in the form

$$\begin{cases} u_t = -(1+\Delta)^2 u - u^3 + \varepsilon u, & \mathbf{x}\in\Omega, \quad t\in(0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x}\in\bar{\Omega} \end{cases} \tag{1.1}$$

with periodic boundary conditions on $\Omega=\prod_{k=1}^{d}(a_k,b_k), d=2$ or 3. Here, $u:\bar{\Omega}\times[0,T]\to\mathbb{R}$ is a periodic density field; $T>0$ is the final time; and $\varepsilon>0$ is a constant related to the temperature. The SH model was originally proposed by Swift and Hohenberg [48] to study the thermal convection of the Rayleigh-Bénard instability at large Prandtl numbers under the Boussinesq approximation. Beyond fluid convection, the SH equation has been pivotal in various pattern formation scenarios in fields such as biology, sociology, crystallography, and others [11,31,44]. Studies on stable patterns and bifurcations for the SH equation can be found in, for example, [42,50].

The SH equation arises as the L^2 gradient flow ($u_t=-\delta E(u)/\delta u$) of the Swift-Hohenberg-type Lyapunov free energy functional [48],

$$E(u)=\int_\Omega \left(\frac{1}{2}u(1+\Delta)^2 u+\frac{1}{4}u^4-\frac{\varepsilon}{2}u^2\right)\mathrm{d}\mathbf{x}. \tag{1.2}$$

It ensures the non-increasing of energy over time, but mass is not conserved. Conversely, employing the H^{-1} gradient flow ($u_t=\Delta\delta E(u)/\delta u$) yields the six-order-in-space phase field crystal (PFC) equation [17], where energy decreases, and mass is conserved. More details could be found in [16]. Because of the broad applications of the SH equation in various research areas and its features such as fourth-order spatial derivative, nonlinearity, and large time scale, we aim to create accurate, efficient, and energy-stable (thermodynamically consistent [22]) numerical schemes. These schemes will facilitate the exploration of diverse stable equilibrium solutions [41] of the SH equation. However, it should be noted that our results are also applicable to the PFC equation.

So far, various energy stable integrators have been developed for SH and PFC-type equations, including nonlinear convex splitting schemes [46, 51, 55], linear convex splitting (stabilization) methods [21, 65], Douglas-Dupont-type regularization [6, 43], stabilizing correction [8, 9], implicit schemes [12, 23, 38, 39, 57, 66], invariant energy quadrant method [40], and scalar auxiliary variable algorithms [58, 67], among others. The linear convex splitting technique [7, 26, 56, 59–62] is notable for effectively integrating nonlinear terms while ensuring stability, attracting attention across various research areas. In particular, the implicit-explicit (IMEX) Euler, exponential time differencing multi-step, and exponential-time-differencing Runge-Kutta (ETDRK) schemes [14, 15, 27, 29] are extensively utilized in solving gradient flow problems [5, 13, 32, 45, 64]. However, achieving energy stability typically necessitates a global Lipschitz assumption on the nonlinearity [20], an a priori ℓ^∞ bound on numerical solutions [26, 65], or a restriction on the time