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Well-Posedness and Regularity Analyses for Nonlocal Nonautonomous System

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Abstract. The time-space nonlocal evolution equations are powerful implementation for modeling anomalous diffusion. In this research, we study the nonlocal nonautonomous reaction-diffusion equation

$$\begin{cases} \partial_t^w u(t,x) = \mathcal{L}u(t,x) + \kappa(t,x)u(t,x), & x \in \mathcal{X}, \quad t \in (0,\infty), \\ u(0,x) = f(x), & x \in \mathcal{X}, \end{cases}$$

where $\mathcal{X}$ is a Lusin space, ∂_t^w is a generalized time fractional derivative, κ is a bounded reaction rate, and $\mathcal{L}$ is an infinitesimal generator in terms of semigroup induced by a symmetric Markov process X. We show that the stochastic representation $u(t,x)$ defined by

$$u(t,x) = \mathbb{E}^x \left[e^{\int_0^t \kappa(r, X_{E_t - E_r}) dE_r} f(X_{E_t}) \right]$$

is the unique mild as well as weak solution. By further analysis, one can get that the above stochastic representation is also the unique strong solution, and the higher spatial and temporal regularity are obtained. In some particular cases, the corresponding dynamical behaviors are displayed.

AMS subject classifications: Primary 35R11, 60K50; Secondary 26A33, 60H30
Key words: Stochastic representation, time fractional nonautonomous equation, well-posedness, regularity.

1 Introduction

Reaction-diffusion systems are frequently used to describe chemical and physics phenomena, population dynamics, and biomedical processes, etc. We refer the readers to [5,6,22,35,36,38,39,42] and references therein for more details. Dynamics in intracellular transportation and translocation of a polymer, also in other fields of natural (or even so-

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cial) sciences, are often non-Brownian. Future prospects for the modeling and simulation of anomalous and nonergodic dynamics are presented in [40]. Biological modeling with nonlocal advection-diffusion equations are discussed in [28]. The macroscopic equations governing the anomalous dynamics, widely observed in nature [15], are often with nonlocal operators, e.g. fractional derivative. One typical case is the Caputo derivative ∂_t^β with fractional order $\beta\in(0,1)$. Owing to particle sticking and trapping phenomena, the equation $\partial_t^\beta u(t,x)=\Delta u(t,x)$ with $u(0,x)=f(x)$ has been widely used to model the anomalous diffusions displaying subdiffusive behavior such as thermal diffusion in fractal media, protein diffusion within cells, and contaminant transport in groundwater [14]. Meerschaert and Scheffer [27, Theorem 5.1] recognized, based on [4], that the stochastic representation $u(t,x)=\mathbb{E}^x[f(B_{E_t})]$ satisfies this equation, where E_t is an inverse β-stable subordinator that is independent of B_t. See stochastic representation for nonlocal-in-time diffusion in [16].

The stochastic representation as a powerful approach displays a relation between the stochastic processes and partial differential equations (PDEs). In the past few decades, many researchers have shown increasing attention and performed extensive study in stochastic representation for PDEs with classical or fractional derivatives [4, 8, 10, 26, 37, 41,45]. However, it seems that there is little research work on the stochastic representation for the PDEs with time-space dependent coefficients. In [17, 25], the authors considered the stochastic representation for the heat equation with external potential by making use of Itô's formula and stopping time. The stochastic representation for the Cauchy problem with external potential was studied in [29] based on an equivalence between the well-posedness of the martingale problem and the well-posedness of the homogeneous Cauchy problem. The stochastic representation that involves power-law or more general waiting time distributions of the underlying random walk was obtained in [8] on the basis of a Lévy process whose Laplace exponent is directly related to the memory kernel.

Throughout this paper, let $\mathcal{X}$ be a Lusin space, which by definition is a topological space that is homeomorphic to a Borel subset of a compact metric space. Let

$$X=\{X_t,t\in[0,\infty);\mathbb{P}^x,x\in\mathcal{X}\}$$

be a time-homogeneous strong Markov process on $\mathcal{X}$ whose sample paths are right continuous and having left limits on $\mathcal{X}\cup\{\partial\}$, where ∂ is an isolated cemetery point outside $\mathcal{X}$ and $X_t=\partial$ for every $t\ge\zeta:=\inf\{t\ge0:X_t=\partial\}$. The transition semigroup $\{P_t\}_{t\ge0}$ of X is defined as

$$P_tf(x):=\mathbb{E}^x[f(X_t)],\quad x\in\mathcal{X},\quad t\ge0$$

for any bounded or nonnegative function f on $\mathcal{X}$ that is extended to $\mathcal{X}\cup\{\partial\}$ by setting $f(\partial)=0$. Here $\mathbb{P}^x$ denotes the probability law of X starting from position x, and $\mathbb{E}^x$ is the mathematical expectation taken under probability law $\mathbb{P}^x$. We assume in addition that the strong Markov process X on $\mathcal{X}$ is ν-symmetric, i.e. for any nonnegative functions f and g on $\mathcal{X}$ and $t>0$,