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Convergent Finite Elements on Arbitrary Meshes, the WG Method

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Abstract. On meshes with the maximum angle condition violated, the standard conforming, nonconforming, and discontinuous Galerkin finite elements do not converge to the true solution when the mesh size goes to zero. It is shown that one type of weak Galerkin finite element method converges on triangular and tetrahedral meshes violating the maximum angle condition, i.e. on arbitrary meshes. Numerical tests confirm the theory.

AMS subject classifications: 65N15, 65N30

Key words: Discontinuous finite element, maximum angle condition, Poisson's equation, triangular grid, tetrahedral grid.

1 Introduction

In one-dimensional finite element computation, the discrete solutions converge to the true solution as long as the size of the largest interval goes to zero, no matter how the mesh points are placed. But it is no longer true in higher dimensions. A famous example was found by Babuška and Aziz [5] that the finite element solution does not converge to the true solution on some patterned triangular meshes (cf. Fig. 1(A)), which violate the maximum angle condition.

The problem can be illustrated by two examples in Calculus. When one approximates a circle by polygons, no matter how the interpolation points are placed, the arc-length of the polygon converges to the circumference of the circle, cf. Fig. 1(E), as long as the longest edge goes to zero. In Fig. 1(B), one does the problem in two dimensions, interpolating a cylinder by linear triangles with a horizontal size h and a vertical size h^2. When $h \to 0$, though the distance between the cylinder and the triangles goes to zero, the total area of triangles converges to a bigger number different from the surface area of the cylin-

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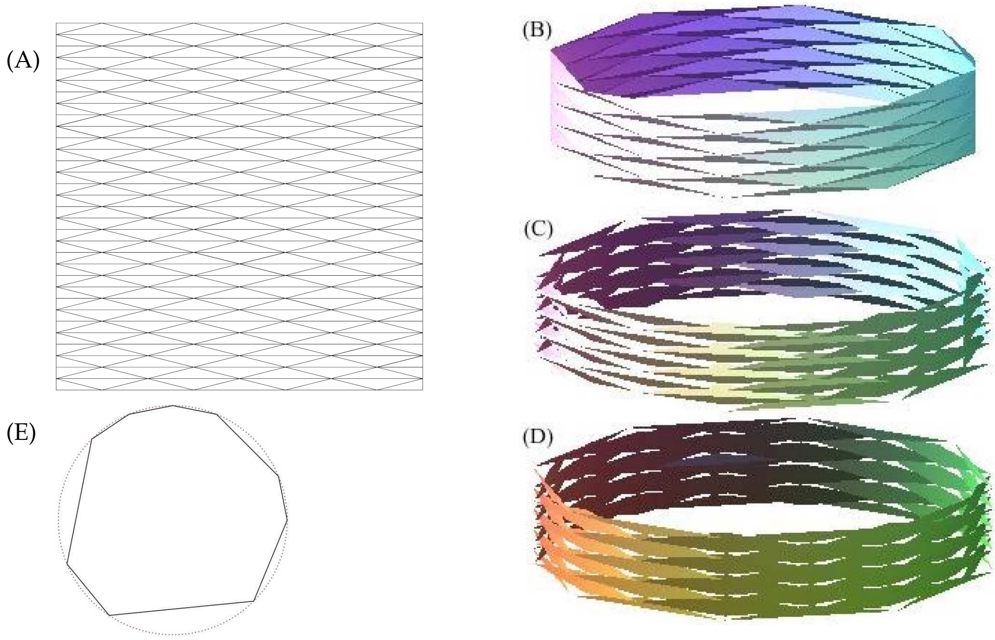

Figure 1: (A) A triangular mesh violates the maximum angle condition, when $h \to 0$. (B) The area of triangles (interpolating at three vertices) converges to different a number. (C) The area of triangles (interpolating at three mid-edge points) converges to different a number. (D) The area of triangles (L^2-projection) converges to that of the cylinder. (E) The arc-length of line segments converges to that of the circle when mesh size goes to zero.

der. One can see that the small triangles keep an angle to the cylinder. In other words, they do not approach the tangent planes locally.

In addition to continuous finite element methods, there is a type of elements [7], called nonconforming finite elements, whose functions are continuous up to the P_{k-1} order, i.e. the jump of two P_k polynomials on two sides of an edge is L^2-orthogonal to the space of P_{k-1} polynomials on the edge. Translating this method to the above cylinder approximation problem, one does the P_1 interpolation at three mid-edge points, instead of the three vertex points, cf. Fig. 1(C). Though we have a slightly better approximation, the problem is not solved as the areas of triangles do not converge to that of the cylinder. Similarly, the non-conforming finite element solutions do not converge to the true solution on such meshes.

But the problem of approximating the cylinder by triangles can be easily solved if we L^2 project the surface to discontinuous P_1 functions on the tangent plane, cf. Fig. 1(D). Then the total area of triangles would converge to the area of the cylinder. Applying the idea to the finite element methods, can we allow arbitrary meshes in the discontinuous Galerkin (DG) and the variances [3, 4, 6] (where totally discontinuous polynomials are used)? Instead of the above L^2-projection, the finite element solution is the global (weak) H^1-projection when solving the Poisson equation. By our primitive numerical tests, none of these discontinuous polynomial methods works, except one.