CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2024-0052

Vol. **6**, No. 4, pp. 666-710
December 2025

Two Methods Addressing Variable-Exponent Fractional Initial and Boundary Value Problems and Abel Integral Equation

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Received 20 November 2024; Accepted 6 March 2025

Abstract. Variable-exponent fractional models attract increasing attentions in various applications, while rigorous mathematical and numerical analysis for typical models remains largely untreated. This work provides general tools to address these models. Specifically, we first develop a convolution method to study the well-posedness, regularity, an inverse problem and numerical approximation for the subdiffusion of variable exponent. For models such as the variable-exponent two-sided space-fractional boundary value problem (including the variable-exponent fractional Laplacian equation as a special case) and the distributed variable-exponent model, for which the convolution method does not apply, we develop a perturbation method to prove their well-posedness. The relations between the convolution method and the perturbation method are discussed, and we further apply the latter to prove the well-posedness of the variable-exponent Abel integral equation and discuss the constraint on the data under different initial values of variable exponent.

AMS subject classifications: 35R11, 45D05, 65M12

Key words: Variable exponent, fractional differential equation, integral equation, mathematical analysis, numerical analysis.

1 Introduction

Variable-exponent fractional models attract increasing attentions in various fields. For instance, the variable-exponent subdiffusion equation has been widely used in the anomalous transient dispersion [52, 53], while the variable-exponent two-sided space-fractional diffusion equation has been applied in, e.g. the turbulent channel flow [48] and the transport through a highly heterogeneous medium [43, 59]. More applications of variable-exponent fractional models can be found in a comprehensive review [51].

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Nevertheless, the theoretical study of variable-exponent fractional initial and boundary value problems as well as the corresponding integral equations is far from well developed. For the subdiffusion model, a typical fractional initial value problem, there exist extensive investigations and significant progresses for the constant-exponent case [6, 19, 26, 29, 30, 33–35, 37, 40, 58], while much fewer investigations for the variable exponent case can be found in the literature. There exist some mathematical and numerical results for the case of space-variable-dependent variable exponent [28, 32, 61]. For the time-variable-dependent case, there are some recent works changing the exponent in the Laplace domain [8, 20] such that the Laplace transform of the variable-exponent fractional operator is available. For the case that the exponent changes in the time domain, the only available results focus on the piecewise-constant variable exponent case such that the solution representation is available on each temporal piece [27, 55]. It is also commented in [27] that the case of smooth exponent remains an open problem. Recently, a local modification of subdiffusion by variable exponent $\alpha(t)$ is proposed in [64], where the techniques work only for the case $\alpha(0)=0$ such that this is mathematically a special case of the variable-exponent subdiffusion model. The general case $0<\alpha(0)<1$ remains untreated.

For fractional and nonlocal boundary value problems, there also exist sophisticated investigations for the constant-exponent case, see e.g. [2, 4, 10, 13–17]. For the variable-exponent case, there are some corresponding theoretical results such as heat kernel estimates for variable-exponent nonlocal operators [7] and well-posedness study for a nonlocal model involving the doubly-variable fractional exponent and possibly truncated interaction [11]. Numerically, some efficient algorithms have been developed for the variable-exponent nonlocal and fractional Laplacian operators [11,56]. Furthermore, a solution landscape approach has been applied for nonlinear problems involving variable-exponent spectral fractional Laplacian [57]. Nevertheless, rigorous analysis of variable-exponent fractional boundary value problems defined via fractional derivatives, e.g. the variable-exponent two-sided space-fractional diffusion equation in the form as [17], is not available in the literature.

For weakly singular integral equations such as the Abel integral equation, extensive results for the constant-exponent models can be found in the literature, see e.g. the books [5, 21]. For the case of variable exponent, there are rare studies. Recently, some works consider the mathematical and numerical analysis for the second-kind weakly singular Volterra integral equation of the variable exponent [36, 65]. For the first-kind integral equations, [63] develops an approximate inverse technique to convert the nonconvolutional Abel integral equation of variable exponent to a second-kind weakly singular Volterra integral equation to facilitate the analysis. For the variable-exponent Abel integral equation of convolutional form, which is a more natural way to introduce the variable exponent, the corresponding study is not available.

The main difficulty for investigating variable-exponent fractional initial or boundary value problems as well as the first-kind variable-exponent Volterra integral equations is that the variable-exponent Abel kernel in these models could not be analytically treated