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An Efficient Iteration Based on Reduced Basis Method for Time-Dependent Problems with Random Inputs

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Abstract. In this paper, we propose an efficient iterative method called RB-iteration, based on reduced basis (RB) techniques, for addressing time-dependent problems with random input parameters. This method reformulates the original model such that the left-hand side is parameter-independent, while the right-hand side remains parameter-dependent, facilitating the application of fixed-point iteration for solving the system. High-fidelity simulations for time-dependent problems often demand considerable computational resources, rendering them impractical for many applications. RB-iteration enhances computational efficiency by executing iterations in a reduced order space. This approach results in significant reductions in computational costs. We conduct a rigorous convergence analysis and present detailed numerical experiments for the RB-iteration method. Our results clearly demonstrate that RB-iteration achieves superior efficiency compared to the direct fixed-point iteration method and provides enhanced accuracy relative to the classical proper orthogonal decomposition (POD) greedy method.

AMS subject classifications: 35B30, 65M12, 65M50, 65R20, 76R99, 78M34

Key words: Reduced order model, time-dependent problems, random inputs, reduced basis, iteration method.

1 Introduction

Parameterized partial differential equations (PDEs) are extensively used in design, control, optimization, and uncertainty quantification. These applications often require a large number of realizations, ranging from thousands to millions. When nonlinear, multi-physics, and time-dependent phenomena are involved, the use of high-fidelity (or full

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order) approximation techniques such as finite element (FE), finite volume (FV), or finite difference (FD) methods imposes substantial computational burdens [1, 3, 7, 8, 21]. Despite the availability of substantial computational resources, repeatedly solving PDEs for various data settings or requiring rapid numerical solutions remains a challenge for high-fidelity numerical techniques. To overcome this computational challenge, many numerical methods have been proposed in recent decades. Spectral stochastic methods [14] have been widely studied for uncertainty propagation in the last two decades. Methods such as L^2 projection [32], Galerkin projection [31], regression [5], and random interpolation [49] aim to construct functional expansions of random solutions using a suitable set of model-independent random variable basis functions. Model order reduction (MOR) offers a very general framework for addressing these challenges.

MOR focuses on active research in fields such as ordinary differential equations (ODEs), PDEs, physics, and engineering [4,39,42,43,48,50]. The success of MOR relies on the assumption that the solution manifold can be embedded in a low-dimensional space, as seen in fields such as fluid dynamics, structural mechanics, and control systems. However, an important class of problems – specifically those involving parametric dynamical systems – often results in a rough solution manifold characterized by slowly decaying Kolmogorov n-widths. This implies that traditional MOR methods are generally ineffective. In this contribution, we try to combine an iteration method with the reduced basis method to construct an efficient and reliable surrogate model of the input-output relationship for parametric time-dependent systems.

The RB method involves sampling the high-fidelity model at various points in both parameter and temporal spaces to construct a snapshot matrix. This matrix is then used to generate the RB through singular value decomposition (SVD). The original high-fidelity model is subsequently projected onto this RB using a Galerkin projection. This process of constructing the snapshot matrix and the model occurs during the offline phase, which is executed only once. In the online phase, the reduced model can be run efficiently for any desired parameter point. RB methods represent a significant class of MOR techniques. Rozza *et al.* [42] offered a comprehensive survey of RB methods and provided numerous examples of their applications. The origins of RB methods can be traced back to the need for more efficient design evaluation in many-query scenarios [13] and for parameter extension in nonlinear structural analysis [2, 35, 36]. Early RB approaches were primarily focused on local approximation spaces and low-dimensional parameter domains. However, subsequent developments aimed to extend these methods to more global and high-dimensional parameter spaces, incorporating stringent error bounds [37] and effective sampling strategies [6, 41]. RB methods have evolved to address both linear and nonlinear elliptic PDEs with affine parameter dependencies [37,47]. In the context of time-dependent problems with random inputs, significant progress has been made using these methods. For instance, Grepl and Patera [16] extended RB methods and the associated a posteriori error estimators, initially developed for elliptic problems, to parabolic problems with affine parameter dependence. They also introduced an adaptive procedure to optimize the sampling set. Quarteroni *et al.* [39] explored various extensions of