CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2024-0016

Vol. **6**, No. 4, pp. 799-841
December 2025

Global Solvability in a Two-Species Keller-Segel-Navier-Stokes System with Sub-Logistic Source

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Received 15 May 2024; Accepted 5 April 2025

Abstract. This paper is concerned with a two-species Keller-Segel-Navier-Stokes model with sub-logistic source in a bounded domain with smooth boundary under no-flux/no-flux/no-flux/Dirichlet boundary conditions. For a large class of cell kinetics including sub-logistic degradation, it is shown that under an explicit condition involving the chemotactic strength and initial mass of cells, the two-dimensional Keller-Segel-Navier-Stokes problem possesses a global and bounded classical solution. In the case with arbitrary superlinear logistic degradation, it is proved that for all suitably regular initial data, the two-dimensional Keller-Segel-Navier-Stokes problem has at least one globally defined solution in an appropriate generalized sense. These results improves and extends the previously known ones.

AMS subject classifications: 35A09, 35D30, 35Q35, 92C17

Key words: Keller-Segel-Navier-Stokes, sub-logistic source, boundedness, generalized solution.

1 Introduction

Chemotaxis is the directed movement of cells or organisms in response to the gradients of concentration of the chemical stimuli and plays essential roles in various biological process such as aggregative patterns of bacteria, slime mold formation, angiogenesis in tumor progression and wound healing [8, 23, 33, 39]. The classical Keller-Segel model proposed by Keller and Segel [26] to the following chemotaxis model:

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$$\begin{cases} n_t = \Delta n - \chi \nabla \cdot (n \nabla c), & x \in \Omega, \quad t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary. The unknown variables $n(x,t)$ and $c(x,t)$ represent the density of cell and the concentration of chemical signal, respectively, $\chi > 0$ is the chemotactic sensitivity. One of striking features of this minimal Keller-Segel model is the possibility of blow-up of solutions in finite or infinite time, which strongly depends on the space dimensional. To be specific, if $N = 1$, the corresponding solution is always globally bounded [38]; if $N = 2$, the total mass $\int_\Omega n_0$ acts a critical value determining whether or not the blow-up of solutions occurs [20, 34]; if $N \geq 3$, the finite-time blow-up of radial solutions may happen with arbitrarily small mass $\|n_0\|_{L^1(\Omega)}$ [54], and the corresponding solution exists globally in time and converges to the constant steady state provided that $\|n_0\|_{L^{N/2}(\Omega)} + \|c_0\|_{L^N(\Omega)}$ is sufficiently small [5,52].

The situation of single-species is rare in nature, mainly in the multi-species case. The following two-species chemotaxis system with Lotka-Volterra competitive kinetics has been considered by many scholars:

$$\begin{cases} n_{1t} = d_1 \Delta n_1 - \nabla \cdot \big(n_1 \chi_1(c) \nabla c\big) + \mu_1 n_1 (1 - n_1 - a_1 n_2), & x \in \Omega, \quad t > 0, \\ n_{2t} = d_2 \Delta n_2 - \nabla \cdot \big(n_2 \chi_2(c) \nabla c\big) + \mu_2 n_2 (1 - n_2 - a_2 n_1), & x \in \Omega, \quad t > 0, \\ c_t = d_3 \Delta c + g(n_1, n_2, c), & x \in \Omega, \quad t > 0. \end{cases} \tag{1.2}$$

In the case $g(n_1,n_2,c) = -c + \alpha n_1 + \beta n_2$ and $\chi_i(c) \equiv \chi_i$ ($i = 1,2$), there are many interesting results. When the two-species do not influence each other in (1.2), in other words, the competitive kinetics term $\mu_1 n_1(1 - n_1 - a_1 n_2)$ and $\mu_2 n_2(1 - n_2 - a_2 n_1)$ are replaced by $\mu_1 n_1(1 - n_1)$ and $\mu_2 n_2(1 - n_2)$, respectively, Negreanu and Tello [35,36] separately claimed that the system (1.2) has unique uniformly bounded solution with $c_t = \varepsilon \Delta c + h(n_1, n_2, c)$, $\varepsilon \in [0,1)$. Mizukami and Yokota [32] removed the restriction of $\varepsilon \in [0,1)$ to obtain similar results. Bai and Winkler [1] proved that the system (1.2) admits a unique global bounded solution for all regular nonnegative initial data in the 2D case. Moreover, the asymptotic behavior of solutions for the system (1.2) was studied. In the 3D case, Lin and Mu [28] proved that if $\mu_1 > 27\chi_1 + 23\chi_2$ and $\mu_2 > 27\chi_2 + 23\chi_1$, the system (1.2) possesses a unique global classical solution which is bounded. For other two-species chemotaxis systems, such as two-species chemotaxis-competition system with two signals and two-species chemotaxis-competition system with loop, we refer the reader to the literature [25, 29, 30, 44]. There are also outstanding works on the case $g(n_1,n_2,c) = -(\alpha n_1 + \beta n_2)c$ in (1.2). Wang *et al.* [50] proved that the system (1.2) admits a unique global bounded classical solution under some conditions. When the Lotka-Volterra competitive kinetics are not taken into account in system (1.2), Zhang and Tao [69] showed that the system possesses a unique global classical solution that is uniformly bounded if

$$\max\{\chi_1, \chi_2\} \|c(x,0)\|_{L^\infty(\Omega)} < \sqrt{\frac{2}{N}} \pi.$$